Partial Fraction Decomposition

Key Terms

- Term
  - An individual set of coefficients and variables serperated by a + or - sign
  - $27x^2 + 5x - 7$ has 3 terms: $27x^2$, $5x$, and -7
- Factor
  - A factor is a term or sum of terms that when multiplied by another factor creates a product
  - $3(x^2 + 4)(x - 5)$ has 3 factors: 3, $(x^2 + 4)$, and $(x - 5)$
- Product:
  - A Product is the answer to a multiplication problem.
  - The equation $2 \times 5 = 10x$ is made up of factors 2 and 5 while the product is 10
- Reducible Factor
  - A Reducible Factor is a factor that can be factored even further than currently expressed.
  - The expression $(x^2-4)(x+2)$ contains the factors (x^2-4) and $(x+2)$. (x^2-4) is a Reducible Factor.
- Irreducible Quadratic Factor
  - An Irreducible Quadratic Factor is a factor in the form of $ax^2 + bx + c$ that can not be reduced using rational or real numbers.
  - The expression $(x^2+x+4)(x^2-4)$ has 2 Quadratic Factors, however, (x^2+x+4) cannot be factored making it an Irreducible Quadratic Factor
- Repeated Factors
  - A Repeated Factor is a factor that has a multiplicity other than one.
  - The expression $x^3(x + 2)^2(x - 2)$ has the factor x repeated 3 times and the factor $(x + 2)$ repeated twice.

Rules For Making PFD

A Partial Fraction Decomposition has 3 rules that must be followed:

1. When the factor is a linear-irreducible factor, it will be re-written as

$$\frac{A}{(ax-r)}$$

2. When the factor is a repeated factor, it will be re-written as

$$\frac{A_1}{(ax-r)} + \frac{A_2}{(ax-r)^2} + \ldots + \frac{A_n}{(ax-r)^n}$$

3. When the factor is an irreducible quadratic, it will be re-written as

$$\frac{Ax+B}{ax^2+bx+c}$$

Example Partial Fraction Decomposition Rule 1

1. Begin with the fraction expression

$$\frac{x+2}{(x+1)(x-1)}$$

2. The denominator has 2 linear factors; $(x+1)$ and $(x-1)$ This means that the expression will be rewritten like this.

$$\frac{A}{(x+1)} + \frac{B}{(x-1)}$$

3. Set that expression equal to the the original expression.

$$\frac{A}{(x+1)} + \frac{B}{(x-1)} = \frac{x+2}{(x+1)(x-1)}$$

4. Multiply both sides by the denominator of the right side of the equation.

$$\frac{A}{(x+1)}((x+1)(x-1) + \frac{B}{(x-1)}(x+1)(x-1) = x+2$$

5. On the A term, $(x+1)$ cancels out. On the B term $(x-1)$ cancels out.

$$A(x-1) + B(x+1) = x+2$$

6. Distribute A and B and then group the terms based on whether or not they have x or not.

$$Ax - A + Bx + B = x+2$$
$$Ax + Bx - A + B = x+2$$
$$(A+B)x + (-A+B) = x+2$$

7. We now have a system of Linear Equations where $(A+B)x = x$ and $(-A+B) = 2$

$$A + B = 1$$
$$-A + B = 2$$
$$2B = 3, B = \frac{3}{2}$$
$$A + \frac{3}{2} = 1, A = -\frac{1}{2}$$

8. So the Partial Fraction Decomposition is is

$$\frac{-\frac{1}{2}}{(x+1)} + \frac{\frac{3}{2}}{(x-1)}$$

Example Partial Fraction Decomposition Rule 2

1. Begin with the fraction expression

$$\frac{x+2}{(x+4)^2}$$

2. Notice that $(x+4)^2$ is an repeated linear factor. Following the rules previously stated, it will be rewritten like this.

$$\frac{A}{(x+4)} + \frac{B}{(x+4)^2}$$

3. Set this equal to the original fraction.

$$\frac{A}{(x+4)} + \frac{B}{(x+4)^2} = \frac{x+2}{(x+4)^2}$$

4. Multiply both sides by the denominator on the right side of the equation

$$\frac{A}{(x+4)}(x+4)^2 + \frac{B}{(x+4)^2}(x+4)^2 = x+2$$

5. Simplify that down until you get to this point.

$$A(x+4) + B = x + 2$$

6. Distribute all of the factors out until you don't have any left.

$$Ax + 4A + B = x + 2$$

7. You can now solve A and B because you have a system of linear equations.

$$Ax = x$$
$$A = 1$$
$$4A + B = 2$$
$$4(1) + B = 2$$
$$B = \frac{1}{2}$$

8. So the Partial Fraction Decomposition for $\frac{x+2}{(x+4)^2}$ is

$$\frac{1}{(x+4)} + \frac{\frac{1}{2}}{(x+4)^2}$$

Example Partial Fraction Decomposition Rule 3

1. Begine with the fraction expression:

$$\frac{x+2}{x^2+4}$$

2. Notice that the denominator x^2+4 does not have any real zeros. That means the polynomial will rewritten like this:

$$\frac{Ax+B}{x^2+4}$$

3. Set this equal to the original fraction

$$\frac{Ax+B}{x^2+4} = \frac{x+2}{x^2+4}$$

4. Multiply both sides by the denominator

$$\frac{Ax+B}{x^2+4}(x^2+4) = x+2$$

$$Ax+B = x+2$$

5. The equation can now be set to a system of linear equations

$$A = 1$$

$$B = 2$$

6. So the expression has a Partial Fraction Decomposition of the following:

$$\frac{x+2}{x^2+4}$$