2023 State Math Competition

Senior Exam

Version A

Instructions:

- Make sure to write your name and mark the version on your answer sheet. Write your school name in the ID space and your grade in the Section space.
- Correct answers are worth 5 points. Unanswered questions will be given 2 points. Incorrect answers will be worth 0 point. This means that it is not in your best interest to guess answers unless you have eliminated some possibilities.
- No materials (textbooks, notes, calculator, internet, etc) allowed.
- Fill in the answers on the answer sheet using a pencil or pen.
- Time limit: 75 minutes.
- When you are finished, please give the exam and any scrap paper to the test administrator.
- Good luck!

1. Assume the following two statements are true

   - Some students are not honest
   - All math majors are honest

   What conclusion can we draw?

   (a) Some students are not math majors
   (b) Some students are math majors
   (c) No math major is a student
   (d) No student is a math major.
   (e) Some math majors are not students

   Correct answers: (a)

   Explanation: There exist dishonest students, and by our second assumption they can not be math majors.

2. If $f(x) = -x$ and $g(x) = \dfrac{x + |x|}{2}$ are defined for all real numbers, then what is the range of the composition $g \circ f(x)$?

   (a) $(-\infty, 0]$
   (b) $(0, \infty)$
   (c) $\{0\}$
   (d) $(-\infty, \infty)$
   (e) $[0, \infty)$

   Correct answers: (e)

   Explanation: We note that $g \circ f(x) = \dfrac{-x + |x|}{2} = \max(-x, 0)$. To see this, note that if $x < 0$ is negative, then $f(x) = -x$ is positive, and hence $|f(x)| + f(x) = 2f(x)$ and so $g \circ f(x) = f(x) = -x > 0$. If $x \geq 0$, then $f(x) = -x \leq 0$ and $|f(x)| + f(x) = 0$ and hence $g \circ f(x) = 0$. The range of $g \circ f$ is $[0, \infty)$.

3. If $\log_{10} x = 10^{\log_{100} 4}$, then x equals

   (a) 100
   (b) 1
   (c) 1000
   (d) 10000
   (e) 10

   Correct answers: (a)

   Explanation: $\log_{100} 4 = \dfrac{\log_{10} 4}{\log_{10} 100} = \dfrac{2 \log_{10} 2}{2} = \log_{10} 2$. Now, $\log_{10} x = 10^{\log_{10} 2} = 2 \Rightarrow x = 10^2 = 100$.

4. The number of one-to-one functions from a set with 3 elements to a set with 6 elements is

   (a) 720
   (b) 210
   (c) 120
   (d) 360
   (e) 240

   Correct answers: (c)

   Explanation: We have $\binom{6}{3} = 20$ ways to pick the three elements of the range. For each of those choices, we have $3! = 6$ ways to map the three elements in the domain to these elements of the range. So the number of such one-to-one functions is $20 \cdot 6 = 120$.

5. We say that two positive integers a and b are relatively prime if $\gcd(a, b) = 1$. For a positive integer n, define $\phi(n)$ to be the number of positive integers less than n and relatively prime to n. For example, $\phi(6) = 2$ since the only numbers less than 6 and relatively prime to 6 are 1 and 5. Find $\phi(\phi(\phi(24)))$.

   (a) 3
   (b) 2
   (c) 1
   (d) 8
   (e) 4

   Correct answers: (b)

   Explanation: Since the numbers relatively prime to 24, less than 24 are: 1,5,7,11,13,17,19,23; $\phi(24) = 8$. Since the numbers relatively prime to 8, less than 8 are: 1,3,5,7; $\phi(8) = 4$. Since the numbers relatively prime to 4, less than 4 are: 1,3; $\phi(4) = 2$. Thus, $\phi(\phi(\phi(24))) = \phi(\phi(8)) = \phi(4) = 2$.

6. A triangle has side lengths 12 and 13. What is the maximum possible area of the triangle?

   (a) 156
   (b) 90
   (c) 66
   (d) 78
   (e) 39

   Correct answers: (d)

   Explanation: We may assume that the triangle is positioned so that the base has length 13 and that the angle between the two given legs is θ. The height of the triangle is $12 \cdot \sin(\theta)$ and its area is $\frac{1}{2} 13 \cdot 12 \cdot \sin(\theta)$. Among all triangles, the area is maximized when $\theta = \frac{\pi}{2}$ and the maximimum area is $\frac{1}{2} 13 \cdot 12 = 6 \cdot 13 = 78$.

7. A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. Then the middle digit is:

   (a) 4
   (b) 5
   (c) 0
   (d) 1
   (e) 3

   Correct answers: (c)

   Explanation: Let xyz be the number when expressed in base 9 and zyx when expressed in base 7. This implies that $81x + 9y + z = 49z + 7y + x$ or $y = 8(3z - 5x)$. Since $0 \leq y \leq 6$ (it appears as a digit in base 7), the integer $n = 3z - 5x$ is zero, otherwise $|8n|$ would be greater than 7. Hence, $y = 0$

8. Given that $|x| + |y| = 3$, what is the minimum possible value of $x^2 + y^2 - 6x + 4y + 16$?

   (a) 7
   (b) 5
   (c) 10
   (d) 8
   (e) 13

   Correct answers: (b)

   Explanation: The graph of $|x| + |y| = 3$ is a square with vertices at $(3, 0)$, $(0, 3)$, $(-3, 0)$, and $(0, -3)$. For example, if $x > 0$ and $y > 0$, then the graph includes the portion of the line $x + y = 3$ in the first quadrant. If $x > 0$ and $y < 0$, then the graph includes the portion of the line $x - y = 3$ in the fourth quadrant, and so on. We want to minimize the value of $x^2 + y^2 - 6x + 4y + 16 = (x - 3)^2 + (y + 2)^2 + 3$. This quantity is 3 more than the square of the distance to the point $(3, -2)$. Dropping a perpendicular from $(3, -2)$ to the line $y = x - 3$, the intersection occurs at point $(2, -1)$. So the minimum possible value is $(2 - 3)^2 + (-1 + 2)^2 + 3 = 5$.

9. An unfair coin has probability p of coming up heads on a single flip. When this coin is flipped three times, the probability that exactly one head will occur is $\frac{1}{8}$. Which is true?

   (a) p is not uniquely determined
   (b) $p = \frac{1}{4}$ is the only solution
   (c) There are no possible solutions for p
   (d) $p = \frac{2}{5}$ is the only solution
   (e) $p = \frac{1}{3}$ is the only solution

   Correct answers: (a)

   Explanation: There are $\binom{3}{1} = 3$ possible orderings of heads, tails, tails. Because the flips are independent, each of these three outcomes has probability $p(1 - p)(1 - p) = p(1 - p)^2$ of occurring. So the probability of getting one heads out of three flips is $3p(1 - p)^2$. The function $f(p) = 3p(1 - p)^2$ has roots at $p = 0$ and $p = 1$ and is positive. We can differentiate it and see that $f'(p) = 3(1 - p)(1 - 3p)$ and so $p = 1/3$ is where the maximum value of $f(1/3) = 4/9$ occurs. Therefore, if the given probability is greater than $4/9$, there will be no solution. If the given probability is $4/9$ there is exactly one solution. If the given probability is less than $4/9$ there are multiple solutions for p. Since $\frac{1}{8} < \frac{4}{9}$, p is not uniquely determined.

10. Select the correct option representing the order of e^π and π^e, and π.

   (a) $e^\pi < \pi < \pi^e$
   (b) $\pi^e < \pi < e^\pi$
   (c) $\pi < \pi^e < e^\pi$
   (d) $\pi < e^\pi = \pi^e$
   (e) $\pi < e^\pi < \pi^e$

   Correct answers: (c)

   Explanation: Consider the function $f(x) = x^{1/x}$ defined for $x > 0$. If you differentiate this function (using logarithmic differentiation), you get that $f'(x) = x^{1/x}\left(\frac{1-\ln x}{x^2}\right)$ and hence f is strictly decreasing on (e, ∞). Therefore, $e^{1/e} > \pi^{1/\pi}$. Taking both sides of the inequality to the power of πe gives $e^\pi > \pi^e$. The inequality $\pi^e > \pi$ is clear since both π and e are > 1.

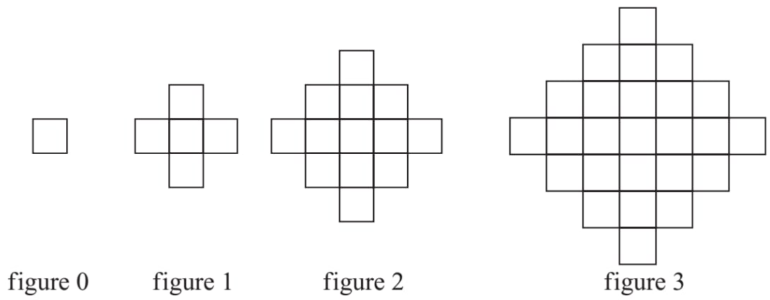
11. Suppose α and β are non-zero solutions to $ax^2 + bx + c = 0$, for some real a, b, c. Which equation necessarily has $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ as solutions?

   (a) $cx^2 - bx + a = 0$
   (b) $ax^2 + bx - c = 0$
   (c) $ax^2 - bx + c = 0$
   (d) $cx^2 + bx + a = 0$
   (e) $-cx^2 + bx - a = 0$

   Correct answers: (d)

   Explanation: Let $f(x) = ax^2 + bx + c$. Then, $g(x) = f(1/x)$ has α^{-1} and β^{-1} as its zeroes. Multiplying with x^2 gives the desired option.

12. Figures 0,1,2 and 3 consist of 1,5,13 and 25 non-overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would be there in figure 100?



   (a) $100^2 + 101^2$
   (b) $100^2 + 102^2$
   (c) $99^2 + 100^2$
   (d) $101^2 + 102^2$
   (e) $99^2 + 101^2$

   Correct answers: (a)

   Explanation: The number of unit squares in the n^{th} figure is the sum of the first n positive odd integers plus the sum of the first $n+1$ positive odd integers. Since the sum of the first k positive odd integers is k^2, the number of unit squares in the n^{th} figure is $n^2 + (n+1)^2$. So the number of unit squares in the 100^{th} figure is $100^2 + 101^2$.

13. The number of polynomials $p(x)$ with integer coefficients such that the curve $y = p(x)$ passes through $(2, 2)$ and $(4, 5)$ is

(a) 2
(b) 1
(c) 0
(d) infinite
(e) more than 2 but finite

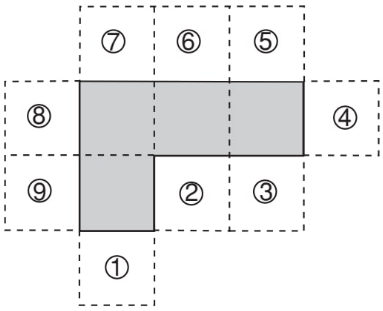
Correct answers: (c)

Explanation: Set $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Then $2 = p(2) = a_n 2^n + a_{n-1}2^{n-1} + \cdots + a_0$. Also, $5 = p(4) = a_n 4^n + a_{n-1}4^{n-1} + \cdots + a_0$. Note then that

$$3 = p(4) - p(2) = a_n(4^n - 2^n) + a_{n-1}(4^{n-1} - 2^{n-1}) + \cdots + a_1(4 - 2)$$

But this gives odd = even, which is not possible. So, no such polynomial exists.

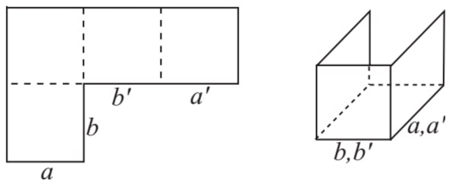
14. The polygon enclosed by the solid lines in the figure below consists of 4 congruent squares joined edge-to edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



(a) 4
(b) 6
(c) 2
(d) 3
(e) 5

Correct answers: (b)

Explanation: If the polygon is folded before the fifth square is attached, then edges a and a' must be joined, as must b and b'. The fifth face of the cube can be attached to any of the six remaining edges.



15. There are 5 strings in a bag, which together have 10 free ends. Reach in and choose two ends randomly, tie them together, and then put them back into the bag. Repeat this step until there no free ends. What is the expected number of closed loops in the bag at the end of the process?

   (a) $\frac{5}{2}$
   (b) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$
   (c) $\left(\frac{10}{9}\right)^4$
   (d) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$
   (e) $\left(\frac{4}{3}\right)^4$

   Correct answers: (d)

   Explanation: There are $\binom{10}{2}$ possible pairs that can be chosen in the first step. Among them 5 pairs increase the number of loops in the bag, otherwise the number of loops is unchanged. The probability that a loop is created in this step is $5/\binom{10}{2} = 1/9$. In the next step, there is a $4/\binom{8}{2} = 1/7$ chance of creating a loop. This continues as $1/5$, $1/3$, and $1/1$. The expected number of loops is therefore $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$.

16. If 2023^{2023} is multiplied out, the units' digits in the final product is

   (a) 9
   (b) 5
   (c) 1
   (d) 3
   (e) 7

   Correct answers: (e)

   Explanation: The final digits for $(2023)^n$ can be listed as follows when n varies:

n	0	1	2	3	4	...
final digits	1	3	9	7	1	...

   We can see that for large values of n these digits repeat in cycles of four.

$$\therefore (2023)^{2023} = (2023)^{4\cdot 505+3} = \left(2023^4\right)^{505} \cdot (2023^3).$$

   The final digit of $\left(2023^4\right)^{505}$ is 1 and the final digit of $(2023)^3$ is 7. The product yields a final digit of $1 \cdot 7 = 7$.

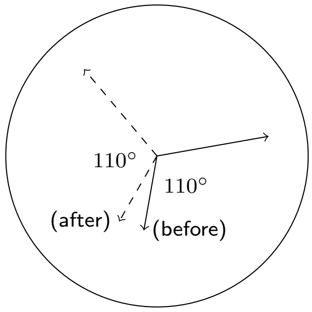
17. A man on his way to dinner shortly after 6 : 00 p.m. observes that the hands of his watch form an angle of 110°. Returning before 7:00 p.m. he notices that again the hands of his watch form an angle of 110°. The number of minutes that he has been away is:

   (a) 45
   (b) 30
   (c) 36
   (d) 42
   (e) 40

   Correct answers: (e)

   Explanation: Let x be the number of degrees the hour hand has moved during the time interval. Then in the same interval, the minute hand has moved $12x$ degrees. But the minute hand has moved $220 + x$ degrees; hence $220 + x = 12x$ or

$x = 20$ and the minute hand has moved $220 + x = 240$ degrees. Since a movement of 6° corresponds to a time interval of 1 minute, the number of minutes that have elapsed is $240/6 = 40$.



18. Find the limiting sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{10^n} = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots$$

(a) $\frac{1}{9}$

(b) $\frac{1}{8}$

(c) infinite

(d) $\frac{17}{12}$

(e) $\frac{10}{81}$

Correct answers: (e)

Explanation: Write the given series as:

$$\begin{aligned} &\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots \\ &\quad + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots \\ &\qquad + \frac{1}{10^3} + \frac{1}{10^4} + \cdots \text{ and so forth.} \end{aligned}$$

These are all geometric series. Let s_1 be the limiting sum of the first-row series, s_2, that of the second-row series, and so forth.

$$\begin{aligned} s_1 &= \frac{1/10}{1-(1/10)} = \frac{1}{9} \\ s_2 &= \frac{(1/10^2)}{1-(1/10)} = \frac{1}{90} \\ s_3 &= \frac{(1/10^3)}{1-(1/10)} = \frac{1}{900} \end{aligned}$$

and so forth. Therefore, the required limiting sum equals the limiting sum of a different geometric series

$$\frac{1}{9} + \frac{1}{90} + \frac{1}{900} + \cdots = \frac{1/9}{1-(1/10)} = \frac{10}{81}$$

19. If $x + \frac{1}{x} = 3$, find $x^5 + \frac{1}{x^5}$.

(a) 123
(b) 224
(c) 243
(d) 171
(e) 111

Correct answers: (a)

Explanation: We have $27 = (x + \frac{1}{x})^3 = x^3 + \frac{1}{x^3} + 3(x + \frac{1}{x}) = x^3 + \frac{1}{x^3} + 9$, so $x^3 + \frac{1}{x^3} = 18$. Similarly, $243 = (x + \frac{1}{x})^5 = x^5 + \frac{1}{x^5} + 5(x^3 + \frac{1}{x^3}) + 10(x + \frac{1}{x}) = x^5 + \frac{1}{x^5} + 5 \cdot 18 + 10 \cdot 3$. Thus $x^5 + \frac{1}{x^5} = 243 - 90 - 30 = 123$.

20. Suppose that x is chosen uniformly at random from $(0, 1)$. What is the probability that $\lfloor \frac{1}{x} \rfloor$ is odd? $\lfloor \frac{1}{x} \rfloor$ is the largest integer less than or equal to $\frac{1}{x}$ and you may use the fact that $\ln(1 + x) = x - \frac{x}{2} + \frac{x}{3} - \cdots$ for $x \in (-1, 1]$.

    (a) $\ln(\sqrt{e})$

    (b) $\ln(1 + \frac{1}{e})$

    (c) $\ln(2)$

    (d) $\ln(\frac{3}{2})$

    (e) $\ln(\frac{4}{3})$

Correct answers: (c)

Explanation: $\lfloor 1/x \rfloor$ is odd iff x is in an interval of the form $\left(\frac{1}{2n}, \frac{1}{2n-1}\right]$ for some $n \geq 1$. The sum of lengths of intervals of this form is

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots = \ln(2).$$

Since the total length of the interval is 1, $\ln(2)$ is the answer.