2023 State Math Competition
Senior Exam
Version A

Instructions:

- Make sure to write your name and mark the version on your answer sheet. Write your school name in the ID space and your grade in the Section space.
- Correct answers are worth 5 points. Unanswered questions will be given 2 points. Incorrect answers will be worth 0 point. This means that it is not in your best interest to guess answers unless you have eliminated some possibilities.
- No materials (textbooks, notes, calculator, internet, etc) allowed.
- Fill in the answers on the answer sheet using a pencil or pen.
- Time limit: 75 minutes.
- When you are finished, please give the exam and any scrap paper to the test administrator.
- Good luck!

1. Assume the following two statements are true

   - Some students are not honest
   - All math majors are honest

   What conclusion can we draw?

   (a) Some students are not math majors
   (b) Some students are math majors
   (c) No math major is a student
   (d) No student is a math major.
   (e) Some math majors are not students

2. If $f(x) = -x$ and $g(x) = \dfrac{x + |x|}{2}$ are defined for all real numbers, then what is the range of the composition $g \circ f(x)$?

   (a) $(-\infty, 0]$
   (b) $(0, \infty)$
   (c) $\{0\}$
   (d) $(-\infty, \infty)$
   (e) $[0, \infty)$

3. If $\log_{10} x = 10^{\log_{100} 4}$, then x equals

   (a) 100
   (b) 1
   (c) 1000
   (d) 10000
   (e) 10

4. The number of one-to-one functions from a set with 3 elements to a set with 6 elements is

   (a) 720
   (b) 210
   (c) 120
   (d) 360
   (e) 240

5. We say that two positive integers a and b are relatively prime if $\gcd(a, b) = 1$. For a positive integer n, define $\phi(n)$ to be the number of positive integers less than n and relatively prime to n. For example, $\phi(6) = 2$ since the only numbers less than 6 and relatively prime to 6 are 1 and 5. Find $\phi(\phi(\phi(24)))$.

   (a) 3
   (b) 2
   (c) 1
   (d) 8
   (e) 4

6. A triangle has side lengths 12 and 13. What is the maximum possible area of the triangle?

   (a) 156
   (b) 90
   (c) 66
   (d) 78
   (e) 39

7. A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. Then the middle digit is:

   (a) 4
   (b) 5
   (c) 0
   (d) 1
   (e) 3

8. Given that $|x| + |y| = 3$, what is the minimum possible value of $x^2 + y^2 - 6x + 4y + 16$?

   (a) 7
   (b) 5
   (c) 10
   (d) 8
   (e) 13

9. An unfair coin has probability p of coming up heads on a single flip. When this coin is flipped three times, the probability that exactly one head will occur is $\frac{1}{8}$. Which is true?

   (a) p is not uniquely determined
   (b) $p = \frac{1}{4}$ is the only solution
   (c) There are no possible solutions for p
   (d) $p = \frac{2}{5}$ is the only solution
   (e) $p = \frac{1}{3}$ is the only solution

10. Select the correct option representing the order of e^π and π^e, and π.

   (a) $e^\pi < \pi < \pi^e$
   (b) $\pi^e < \pi < e^\pi$
   (c) $\pi < \pi^e < e^\pi$
   (d) $\pi < e^\pi = \pi^e$
   (e) $\pi < e^\pi < \pi^e$

11. Suppose α and β are non-zero solutions to $ax^2 + bx + c = 0$, for some real a, b, c. Which equation necessarily has $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ as solutions?

   (a) $cx^2 - bx + a = 0$
   (b) $ax^2 + bx - c = 0$
   (c) $ax^2 - bx + c = 0$
   (d) $cx^2 + bx + a = 0$
   (e) $-cx^2 + bx - a = 0$

12. Figures 0,1,2 and 3 consist of 1,5,13 and 25 non-overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would be there in figure 100?

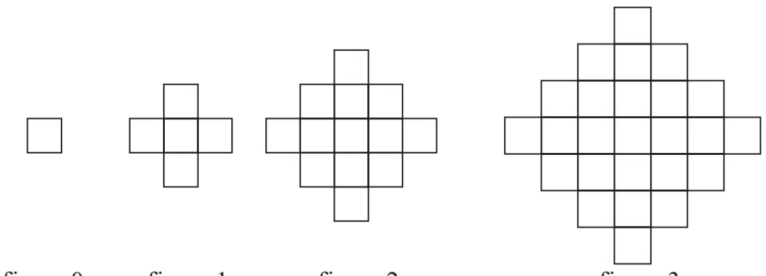


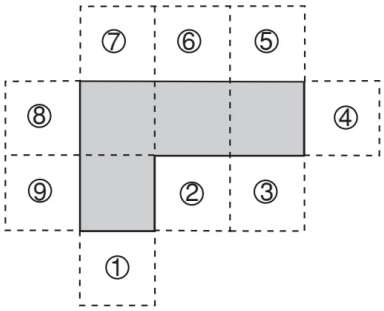
figure 0 figure 1 figure 2 figure 3

   (a) $100^2 + 101^2$
   (b) $100^2 + 102^2$
   (c) $99^2 + 100^2$
   (d) $101^2 + 102^2$
   (e) $99^2 + 101^2$

13. The number of polynomials $p(x)$ with integer coefficients such that the curve $y = p(x)$ passes through $(2, 2)$ and $(4, 5)$ is

(a) 2
(b) 1
(c) 0
(d) infinite
(e) more than 2 but finite

14. The polygon enclosed by the solid lines in the figure below consists of 4 congruent squares joined edge-to edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



(a) 4
(b) 6
(c) 2
(d) 3
(e) 5

15. There are 5 strings in a bag, which together have 10 free ends. Reach in and choose two ends randomly, tie them together, and then put them back into the bag. Repeat this step until there no free ends. What is the expected number of closed loops in the bag at the end of the process?

(a) $\frac{5}{2}$
(b) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$
(c) $\left(\frac{10}{9}\right)^4$
(d) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$
(e) $\left(\frac{4}{3}\right)^4$

16. If 2023^{2023} is multiplied out, the units' digits in the final product is

(a) 9
(b) 5
(c) 1
(d) 3
(e) 7

17. A man on his way to dinner shortly after 6 : 00 p.m. observes that the hands of his watch form an angle of 110°. Returning before 7:00 p.m. he notices that again the hands of his watch form an angle of 110°. The number of minutes that he has been away is:

(a) 45
(b) 30
(c) 36
(d) 42
(e) 40

18. Find the limiting sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{10^n} = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots$$

(a) $\frac{1}{9}$

(b) $\frac{1}{8}$

(c) infinite

(d) $\frac{17}{12}$

(e) $\frac{10}{81}$

19. If $x + \frac{1}{x} = 3$, find $x^5 + \frac{1}{x^5}$.

(a) 123
(b) 224
(c) 243
(d) 171
(e) 111

20. Suppose that x is chosen uniformly at random from $(0, 1)$. What is the probability that $\lfloor \frac{1}{x} \rfloor$ is odd? $\lfloor \frac{1}{x} \rfloor$ is the largest integer less than or equal to $\frac{1}{x}$ and you may use the fact that $\ln(1 + x) = x - \frac{x}{2} + \frac{x}{3} - \cdots$ for $x \in (-1, 1]$.

(a) $\ln(\sqrt{e})$

(b) $\ln(1 + \frac{1}{e})$

(c) $\ln(2)$

(d) $\ln(\frac{3}{2})$

(e) $\ln(\frac{4}{3})$