2023 State Math Competition

Junior Exam

Version A

Instructions:

- Make sure to write your name and mark the version on your answer sheet. Write your school name in the ID space and your grade in the Section space.
- Correct answers are worth 5 points. Unanswered questions will be given 2 points. Incorrect answers will be worth 0 point. This means that it is not in your best interest to guess answers unless you have eliminated some possibilities.
- No materials (textbooks, notes, calculator, internet, etc) allowed.
- Fill in the answers on the answer sheet using a pencil or pen.
- Time limit: 75 minutes.
- When you are finished, please give the exam and any scrap paper to the test administrator.
- Good luck!

1. The wholesale price of an item was increased by 25% when the item arrived in the store. After a few weeks, the item was put on sale for the wholesale price. By what percent was the price reduced for the sale?

   (a) 15
   (b) 20
   (c) 16
   (d) 36
   (e) 25

2. Every angle of a certain triangle is acute and the size of the largest angle is 5 times that of the smallest angle. If all of the angles have integer degree measurements, then the smallest angle is

   (a) $11°$
   (b) $17°$
   (c) $14°$
   (d) $18°$
   (e) $12°$

3. The number 2532645918 is divisible by

   (a) both 3 and 11 but not 4
   (b) neither 3 nor 11 nor 4
   (c) both 3 and 4 but not 11
   (d) 3 but neither 11 nor 4
   (e) both 4 and 11 but not 3

4. How many pairs (b, c) exist such that $4x^2 + bx + c$ has a repeated real root, when b and c are each allowed to be positive integers less than or equal to 100?

   (a) 36
   (b) 4
   (c) 100
   (d) 50
   (e) 10

5. A regular hexagon and an equilateral triangle have equal circumferences. If you divide the area of the hexagon by the area of the triangle, what number do you get?

   (a) 2/1
   (b) 4/3
   (c) 8/3
   (d) 4/1
   (e) 3/2

6. How many integers x exist such that $\dfrac{x^3 + 3x^2 - x - 10}{x + 2}$ is an integer?

   (a) 2
   (b) 6
   (c) 4
   (d) 0
   (e) 8

7. Let $a \vee b$ represent the operation on two numbers, a and b, which selects the larger of the two numbers, with $a \vee a = a$. Let $a \wedge b$ represent the operation which selects the smaller of the two numbers, with $a \wedge a = a$. Which of the following three rules are/is valid for all a, b, c?

   (1) $a \vee b = b \vee a$,

   (2) $a \vee (b \vee c) = (a \vee b) \vee c$,

   (3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

   (a) (1) and (2) only
   (b) (1) and (3) only
   (c) all three
   (d) (1) only
   (e) (2) only

8. There are 24 coins in a bag, each of which is a dime or a quarter. There are an even number of dimes in the bag. If half of the dimes were removed and replaced with the same number of quarters, the value in the bag would increase by \$1.05. How much are the coins in the bag worth?

   (a) \$3.70
   (b) \$3.90
   (c) \$4.20
   (d) \$4.60
   (e) \$3.30

9. What is the units digit of 3^{2023}?

   (a) 3
   (b) 4
   (c) 1
   (d) 9
   (e) 7

10. Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk, expressed in square kilometers and rounded to the nearest whole number?

    (a) 40
    (b) 42
    (c) 24
    (d) 39
    (e) 27

11. Define $n! = 1 \cdot 2 \cdots n$ for all integers $n \geq 1$. What is $\sum_{k=1}^{100} k \cdot k!$.

    (a) $101! - 1$
    (b) $101! + 100!$
    (c) $101! + 101$
    (d) $101!$
    (e) $101! - 101$

12. The Jones family is going on a summer vacation. They will depart on the last day of the month. If you multiply the following four numbers:

    - the day of their departure
    - the month of their return
    - the number of pets they own
    - the length of their vacation,

    you get $14384 = 2^4 \cdot 31 \cdot 29$. How many pets does the Jones family own?

    (a) 3
    (b) 5
    (c) 2
    (d) 4
    (e) 1

13. A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

(a) $\frac{1}{2}$

(b) $\frac{2}{5}$

(c) $\frac{3}{10}$

(d) $\frac{7}{10}$

(e) $\frac{3}{5}$

14. The polynomial $f(x) = (2+x)^n$ can be expanded as $f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$ where n is a positive integer. If $\sum_{j=0}^{n} c_j = 81$, then the largest coefficient c_j of f is

(a) 24
(b) 32
(c) 64
(d) 16
(e) 36

15. Let $a = 3^{13} \cdot 5^7 \cdot 7^{20}$. How many positive integers k have the property that k divides a and 105 divides k?

(a) 1824
(b) 1820
(c) 1771
(d) 2036
(e) 1368

16. Eight players are participating in a chess tournament. They each play one another once, getting 1 point for a win, $\frac{1}{2}$ for a draw, and 0 for a loss. This is their ranking at the end of the tournament:

    1. Amelia
    2. Bill
    3. Clyde
    4. Diane
    5. Ellie
    6. Faye
    7. Gertrude
    8. Hayden

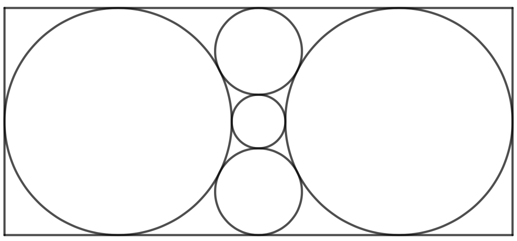
    At the end of the tournament, no two participants have the same number of points. Bill's score is equal to the sum the last four players' scores. How did the game between Diane and Ellie go?

    (a) we can rule out Ellie's victory, but both Diane's win and a draw are possible
    (b) not enough information to determine
    (c) It was a draw
    (d) Ellie won
    (e) Diane won

17. The number of solutions to $2^{2x} - 3^{2y} = 55$, in which x and y are integers, is:

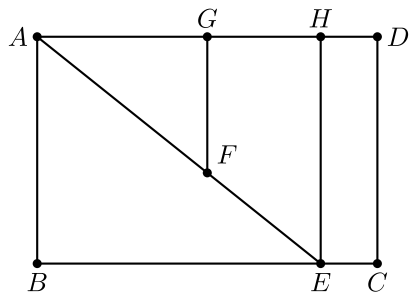
    (a) more than three, but finite
    (b) three
    (c) zero
    (d) two
    (e) one

18. The shorter side of the rectangle pictured below has length 1. How long is the longer side?



    (a) $\sqrt{3}$
    (b) 2
    (c) $\sqrt{6}$
    (d) $\sqrt{5}$
    (e) $\sqrt{2}$

19. In rectangle $ABCD$, point E is on $\overline{BC}$ so that $AE = 7$, and point F is on $\overline{AE}$ so that $AF = 4$. G and H are on $\overline{AD}$ so that $\overline{FG} \parallel \overline{EH} \parallel \overline{AB}$, and $GH = 2$. What is the length AB?



(a) 5
(b) $\frac{9}{2}$
(c) $\frac{7\sqrt{5}}{3}$
(d) $3\sqrt{3}$
(e) $\frac{10\sqrt{2}}{3}$

20. Consider 8-step paths that start at $(-2,-2)$ and end at $(2,2)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths are there that do not include $(0,0)$?

(a) 48
(b) 34
(c) 64
(d) 24
(e) 28