State Math Contest 2024 Senior Level Solutions

Instructions:

- Calculators, cell phones and other computational devices are not permitted (you can only use pens, pencils and paper to work on your answers, and then mark your answers with a number two pencil on the answer sheet).
- Correct answers are worth 5 points. Unanswered questions will be given 1 point. Incorrect answers will be worth 0 points. *This means that it will not, on average, increase your score to guess answers randomly.*
- Fill in the answers on the answer sheet using a number two pencil.
- Time limit: 120 minutes.
- When you are finished, please give the exam and any scratch paper to the test administrator.
- Good luck!

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1. Solve $\log x + \log(x+15) = 2$.

   A. $x = 5$
   B. $x = 5$ or $x = 20$
   C. $x = 5$ or $x = -20$
   D. $x = 15$
   E. $x = 10$

Solution: $\log x + \log(x+15) = 2 \implies \log(x)(x+15) = 2$, converting it to exponential form yields $x^2 + 15x = 10^2$ or $x^2 + 15x - 100 = 0$, which factors to $(x-5)(x+20) = 0$ which has solutions $5, -20$. However, -20 is not in the domain of $\log(x)$, so $x = 5$ is the only solution.

2. Given that $i^2 = -1$, find $(1 + i\sqrt{3})^{10}$.

   A. $-512 - 512\sqrt{3}i$
   B. $-256 + 265\sqrt{3}i$
   C. $-512\sqrt{3} - 512i$
   D. $-128 + 128\sqrt{6}i$
   E. $1024 + 1024\sqrt{3}i$

Solution: Since $\sqrt{1+3} = 2$ and $\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$, we have $(1+i\sqrt{3})^{10} =$ $\left(2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)\right)^{10} = 2^{10}\left(\cos\left(\frac{10\pi}{3}\right) + i\sin\left(\frac{10\pi}{3}\right)\right) = 1024\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) =$ $-512 - 512\sqrt{3}i$.

3. Let $f(x-1)+f(x+1)=f(x)$ for all integers x. If $f(0)=2$ and $f(1)=4$ then find $f(2022)+f(2023)+f(2024)$.

   A. 12

   B. 8

   C. 337

   D. 0

   E. 4

> **Solution:** First, observe that for any integer n (setting $n=x+1$) it is true that $f(n)=f(n-1)-f(n-2)$, so $f(n+1)=f(n)-f(n-1)$, which means that $-f(n-2)=f(n+1)$ and therefore $f(n+3)=-f(n)$ and $f(n+6)=-f(n+3)=f(n)$.
>
> Next, observe that $f(2)=4-2=2$, $f(3)=-2$, $f(4)=-4$, $f(5)=-2$ and then the values of f repeat starting with $f(6)=f(0)=2$. The remainder when 2022 is divided by 6 is zero, which means that $f(2022)=f(0)=2$, $f(2023)=4$ and $f(2024)=2$, so the sum is equal to 8.

4. Evaluate the sum $\dfrac{1}{(1)(2)}+\dfrac{1}{(2)(3)}+\dfrac{1}{(3)(4)}+...+\dfrac{1}{(19)(20)}$.

   A. $\dfrac{19}{20}$

   B. $\dfrac{19!-1}{20!}$

   C. $\dfrac{121}{20!}$

   D. 0.96

   E. 1.04

> **Solution:** This is $\displaystyle\sum_{n=1}^{19}\frac{1}{(n)(n+1)}$ which can be written as $\displaystyle\sum_{n=1}^{19}\frac{1}{n}-\frac{1}{n+1}$. The terms in this sum cancel except the first and the last, giving $1-\dfrac{1}{20}=\dfrac{19}{20}$.

5. Let C be the circle of radius 24 centered at the origin in the plane. Let R be a radius of C (a line segment from the origin to a point of C). Let P be the perpendicular bisector of R, which separates the plane into sides U and V, where U contains the origin. Find the length of the arc A which consists of the points of C which are not in U.

   A. 8π

   B. 40

   C. 0

   D. 6π

   E. 16π

Solution: If we have P be the vertical line through $x = 12$ and R be the line segment from the origin to $(24, 0)$ then we see that the portion of the circle further from the origin than P is the arc A between the points on C with x coordinate 12. In particular, if θ is the angle between the positive x-axis and the radius from the origin to the point of intersection of P and C in the first quadrant then $24\cos(\theta) = 12$, so $\theta = \frac{\pi}{3}$. This means that the angle subtended by A is $\frac{2\pi}{3}$, so the length of A is $24\left(\frac{2\pi}{3}\right) = 16\pi$.

6. How many integers between 1 and $10,000$ are divisible by each of the numbers $2, 3, 4, 5, 6, 7, 8, 9$?
   - A. 0
   - B. 1
   - **C. 3**
   - D. 5
   - E. 28

Solution: An integer which is divisible by nine and eight is also divisible by six, two, three, and four, which means that we can rephrase the problem to be "how many integers are divisible by 8, 9, 7, and 5 between 1 and 10,000?" Such an integer must have the form $8^i 9^j 5^k 7^l(m)$ for positive integers i, j, k, l, m since the prime factorizations of $8, 9, 7$, and 5 have no common factors. Thus, any number divisible by $8, 9, 7$ and 5 must be at least $(9)(8)(7)(5) = 2520$. Raising the power of any of $8, 9, 7$, or 5 to two gives a product that is at least $(9)(8)(7)(5^2) = 12600 > 10000$. Hence, the numbers divisible by $8, 9, 7$, and 5 which are less than 10,000 have the form $2520m$ for a positive integer m. We see that $2520(4) > 10,000$, so the only options are $2520, 2520(2), 2520(3)$.

7. Let $x^2 + y^2 - 7x + 4y + 13 = 0$. What is the smallest possible value of $3x - 2y$?
   - A. 4
   - B. $2\sqrt{3}$
   - C. -4
   - **D. 8**
   - E. There is no smallest value

Solution: Moving $3x - 2y$ to one side of the equation gives us that $3x - 2y = x^2 - 4x + y^2 + 2y + 13$. Completing the square tells us $3x - 2y = (x - 2)^2 + (y + 1)^2 + 8$. The right side is smallest when $x = 2$ and $y = -1$, taking on a value of 8. Since this is a solution to the equation (since $3(2) - 2(-1) = 8$) this must be the minimum value.

8. A survey team places a 24-foot tall vertical pole on top of a (rather short) hill. Someone standing some unknown distance horizontally from the hill measures the angle of elevation to the bottom of the pole to be $30°$ and the angle of elevation to the top of the pole to be $45°$. Find the height of the hill.

   A. 48 feet
   B. $48(\sqrt{2}-1)$ feet
   C. $\dfrac{24(\sqrt{3}-1)}{5}$ feet
   D. $12(\sqrt{3}+1)$ **feet**
   E. 24 feet

> **Solution:** Let A be the point at which the observer is standing, let B be the point at the top of the pole, and C be the point at the bottom of the pole. Let D be the point at height zero on the line through B and C. Then we are given that $\angle BAD = 45°$ and $\angle CAD = 30°$, and $BC = 24$. This means that $\angle BAC = 15°$ and $\angle ABD = 45°$, so $\angle ACB = 120°$. We also know $\angle ACD = 60°$.
>
> Applying the law of sines, we see that $\dfrac{24}{\sin(15°)} = \dfrac{AC}{\sin(45°)}$, so $AC = 12\sqrt{2}\csc(15°)$. The height of the hill is $CD = AC\sin(30°) = \dfrac{1}{2}(AC) = 6\sqrt{2}\csc(15°)$.
>
> We use the difference of angles formula to find $\sin(15°) = \sin(45°-30°) = \sin(45°)\cos(30°) - \cos(45°)\sin(30°) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$. Thus, $h = 6\sqrt{2}\left(\dfrac{4}{\sqrt{6}-\sqrt{2}}\right) = 24\sqrt{2}\left(\dfrac{\sqrt{6}+\sqrt{2}}{6-2}\right) = \dfrac{48\sqrt{3}}{4} + \dfrac{48}{4} = 12(\sqrt{3}+1)$.

9. How many different "words" can be made out of the letters of SASSAFRAS, if a "word" means any string of all nine letters? For example, FRAAASSSS would be a word.

   A. 62
   B. 2048
   C. 2520
   D. 362880
   E. 15120

> **Solution:** This is a permutation on a set of nine objects in which a set of three and a set of four are indistinguishable, so the number is $\dfrac{9!}{(4!)(3!)} = 2520$.

10. Let $\sin(x) - \cos(x) = \dfrac{1}{3}$. Find $\sin(2x)$.

   A. $\dfrac{5}{9}$
   B. $\dfrac{8}{9}$
   C. $\dfrac{7}{9}$
   D. $\dfrac{2}{9}$

E. $\frac{4}{9}$

Solution: Squaring both sides, we get $\sin^2(x) - 2\sin(x)\cos(x) + \cos^2(x) = \frac{1}{9}$, so $1 - 2\sin(x)\cos(x) = 1 - \sin(2x) = \frac{1}{9}$, so $\sin(2x) = \frac{8}{9}$.

11. Let r and a be positive real numbers so that a disk of radius r is small enough to fit inside an equilateral triangle T of side length a. Find the area of the region enclosed by T consisting of all points which are not enclosed by any circle of radius r which is enclosed by T.

A. $(a^2 - r^2)(3\sqrt{3} - \pi)$

B. $\frac{a^2}{r^2}(6\sqrt{2} - 2\pi)$

C. $r^2(\frac{\sqrt{3}}{4} - \frac{\pi}{12})$

D. $r^2(3\sqrt{3} - \pi)$

E. $r^2(\pi - \sqrt{3})$

Solution: We first verify that the side length of a does not affect the solution. The only points enclosed by T that would not be covered by a disk of radius r enclosed by T are near the corners, and the area excluded near the corners is determined by the angle at the vertex (which is $\pi/3$) and the radius r of the circle. Let C be a circle of radius r with center O which is tangent to two edges of T with common vertex D of T. Let W be the quadilateral whose vertices are O, D, and the points A, B of intersection of C and T. The area enclosed by W can be obtained by splitting W into two congruent triangles $\triangle ODA$ and $\triangle ODB$. The height of such a triangle is r and its base has length $\cot\left(\frac{\pi}{6}\right) r = \sqrt{3}r$. Hence, the area enclosed by W is $\sqrt{3}r^2$. The interior angles of W are $\angle ADB = \frac{\pi}{3}$, $\angle DBO = \angle DAO = \frac{\pi}{2}$, and $\angle AOB = \frac{2\pi}{3}$. Since $\angle AOB = \frac{2\pi}{3}$, the area of the circular sector in the circle C from segment $\overline{OA}$ to segment $\overline{OB}$ is $\frac{\pi r^2}{3}$. Thus, the area within W not enclosed by C is $\frac{r^2}{3}(3\sqrt{3} - \pi)$. Since the same area is not enclosed by any circle of radius r near each of the three vertices of the triangle, the total area not covered by disks of radius r enclosed by T is $r^2(3\sqrt{3} - \pi)$.

12. Let a, b, c, x, y, z be non-zero complex numbers so that $a = \frac{b+c}{x+1}$, $b = \frac{c+a}{y+1}$, $c = \frac{a+b}{z+1}$. If $xy + yz + zx = 100$ and $x + y + z = 50$, find the value of xyz.

A. -96

B. -94

C. -82

D. -86

E. -92

Solution: Adding one to both sides of $\dfrac{b+c}{a} = x+1$ we have $\dfrac{a+b+c}{a} = x+2$, so $\dfrac{a}{a+b+c} = \dfrac{1}{x+2}$. Likewise, we are able to get that $\dfrac{b}{a+b+c} = \dfrac{1}{y+2}$ and $\dfrac{c}{a+b+c} = \dfrac{1}{z+2}$. Adding these gives $\dfrac{1}{x+2} + \dfrac{1}{y+2} + \dfrac{1}{z+2} = 1$. Thus,

$$\frac{xy+yz+zx+4(x+y+z)+12}{xyz+2(xy+yz+zx)+4(x+y+z)+8} = 1$$

Hence, $\dfrac{312}{xyz+408} = 1$, so $xyz = -96$.

13. Which of the following is the graph of $x^2 + 2\sqrt{3}xy + 3y^2 + 6x + 16 = 0$?

    A. A parabola

    B. An ellipse

    C. A hyperbola

    D. A line

    E. None of these

Solution: The graph of a non-degenerate conic section $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is a parabola if $B^2-4AC=0$, an ellipse if $B^2-4AC<0$ and a hyperbola if $B^2-4AC>0$. In this case, $B^2-4AC = 12-12 = 0$, so we have a parabola. We can see the conic is non-degenerate by plotting a few points or by checking that $\det \begin{bmatrix} 1 & \sqrt{3} & 3 \\ \sqrt{3} & 3 & 0 \\ 3 & 0 & 16 \end{bmatrix} \neq 0$.

14. Let S consist of all numbers which can be obtained by starting at 0 and then applying the functions $f(x) = x+1$ and $g(x) = \dfrac{x}{2}$ finitely many times (in any possible function composition order; for example, $g(f(f(g(g(0)))))$ would be one of the points of S).

    Which of the following is true?

    (I) If n is in S then n^2 is in S.
    (II) If n is in S then $n-1$ is in S.
    (III) If n is in S then n^2-n is in S.

    A. (I) only

    B. (I) and (II) only

    C. (I), (II), and (III)

    D. (II) and (III) only

    E. None of these are true

Solution: Since neither operation outputs a negative number from a non-negative input, no combination of the operations starting at zero can end in a negative number, so $0-1 = -1 \notin S$, so II is false. Likewise, $\dfrac{0+1}{2} = g(f(0)) \in S$, and $\dfrac{1}{2} - 1 = -\dfrac{1}{2} \notin S$, so III is false.

We show that S consists of all non-negative integers divided by all non-negative powers of two. First, for any non-negative integers n, m if we apply f n times to zero we get n, and if we then apply g m times we get an output of $\frac{n}{2^m}$, so all non-negative integers divided by all non-negative powers of two are in S.

Next, we show by induction that all elements of S are of the form $\frac{n}{2^m}$ for non-negative integers n and m. We induct on the number of operations applied to zero. If we apply one operation to 0 we get 0 or 1, which are both of the form described. Suppose all numbers obtained after performing k operations of type f or g gives a number of the form $\frac{n}{2^m}$. Then applying f an additional time to the result would give a number of the form $\frac{n}{2^m}+1 = \frac{2^m+n}{2^m}$ which is a non-negative integer over a non-negative power of two. Alternately, applying g an additional time would give $\frac{n}{2^{m+1}}$ which is a non-negative integer over a non-negative power of two. By induction, then, we see that $S = \left\{\frac{n}{2^m} \in \mathbb{Q} | n, m, \in \{0\} \cup \mathbb{N}\right\}$, the non-negative dyadic rationals. In particular, if $x = \frac{n}{2^m} \in S$ then $x^2 = \frac{n^2}{2^{2m}}$, where $\frac{n^2}{2^{2m}}$ is a non-negative integer over a non-negative power of two and is therefore in S, which implies that I is true.

15. Let $f(x) = e^{3x+1}$. Find $\lim_{x\to 0} \frac{f(f(x)) - f(e)}{x}$.

    A. $3e^{e^e}$
    B. $\frac{1}{e}$
    C. $9e^{3e+2}$
    D. $9e^{6e^2}$
    E. 0

Solution: First, we note that $f(0) = e$ and $f'(x) = 3e^{3x+1}$. For the function $f \circ f$ we have $(f \circ f)(0) = f(e)$, which means that $\lim_{x\to 0} \frac{f(f(x)) - f(e)}{x} = \lim_{x\to 0} \frac{f(f(x)) - f(f(0))}{x - 0}$ is the derivative of $f \circ f$ at the point 0. Using the chain rule this derivative is $f'(f(0))f'(0) = (3e^{3e+1})(3e^1) = 9e^{3e+2}$.

16. Find the smallest natural number m so that $\sum_{n=1}^{m} \lfloor \log_{10}(n) \rfloor > 2024$. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x).

    A. 10^{2024}
    B. 1035
    C. 899
    D. 1190
    E. 1044

Solution: The greatest integer less than or equal to $\log_{10} n = 0$ if $1 \le n \le 9$. For $10 \le n \le 99$, $\lfloor \log_{10}(n) \rfloor = 1$, so $\sum_{n=1}^{99} \lfloor \log_{10}(n) \rfloor = 90$. If $100 \le n \le 999$ then $\lfloor \log_{10}(n) \rfloor = 2$, so $\sum_{n=1}^{999} \lfloor \log_{10}(n) \rfloor = 90 + 2(900) = 1890$. If $1000 \le n \le 9999$, $\lfloor \log_{10}(n) \rfloor = 3$, and $2024 - 1890 =$

134. The first integer k so that $3k > 134$ is 45, which means that the first integer m so that $\sum_{n=1}^{m} \lfloor \log_{10}(n) \rfloor > 2024$ is $999 + 45 = 1044$.

17. Let $f(x) = ax^2 + bx + c$ be a quadratic function so that $f(1) + f(2) + ... + f(n) = n^3$ for every positive integer n. What is abc?
    - A. -6
    - B. -8
    - **C.** -9
    - D. -10
    - E. -13

Solution: We plug in integers $1, 2, 3$ to give us equations

$$\begin{aligned} f(1) &= a + b + c = 1^3 = 1 \\ f(1) + f(2) &= 5a + 3b + 2c = 2^3 = 8 \\ f(1) + f(2) + f(3) &= 14a + 6b + 3c = 3^3 = 27 \end{aligned}$$

Subtracting twice the first equation from the second and three times the first equation from the third would give us $3a + b = 6$ and $11a + 3b = 24$. Subtracting three times the first of these from the second would give $2a = 6$ so $a = 3$. Substituting gives $b = -3$ and $c = 1$. The product is therefore $(3)(-3)(1) = -9$.

It is worth noting that this does actually give a polynomial satisfying the requirements in question since $\sum_{i=1}^{n} 3n^2 - \sum_{i=1}^{n} 3n + \sum_{i=1}^{n} 1 = 3\left(\frac{n(n+1)(2n+1)}{6}\right) - 3\left(\frac{n(n+1)}{2}\right) + n = n^3$.

18. Let $f(x)$ be a degree four polynomial so that $f(x) < 0$ for all real x. Which of the following must be true about $g(x) = f(x) + f'(x) + f''(x) + f'''(x) + f^{(4)}(x)$?
    - A. $g'''(x) = 0$ for all x.
    - B. $g(x)$ has degree five.
    - C. $g(x) > 0$ for some real x
    - **D.** $g(x) < 0$ **for all real** x
    - E. None of these need to be true.

Solution: First, observe that since f is a degree four polynomial, its graph's end behavior in both directions corresponds to the coefficient of the highest power term, which must be negative. Each of the derivatives of f has lower degree than f, so g is degree four and has the same leading coefficient as f, which is negative. The maximum value of g must occur at a point c where $g'(c) = 0 = f'(c) + f''(c) + f'''(c) + f^{(4)}(c) + f^{(5)}(c) = 0$, and $f^{(5)}(c) = 0$ since f is degree four. Hence, $g(c) = f(c) < 0$. Thus, $g(x) < 0$ for all real x.

We have already noted that (B) and (C) are false. A degree four polynomial's third derivative is degree one and can't be identically zero. Hence, none of (A), (B) or (C) can be true.

19. What is the number of positive integer solutions (x, y) of the equation $x^2+y^2 = 4x+4y+5-2xy$?

    A. 0

    B. 1

    C. 3

    D. 4

    E. Infinitely many

Solution: We write this as $x^2+y^2+2xy-4(x+y)-5 = 0$, which is the same as $(x+y)^2 - 4(x+y)-5 = 0$. If we treat $x+y$ as a variable this factors as $((x+y)+1)((x+y)-5) = 0$, so $(x+y) = 5$. The possible solutions are thus $(1, 4), (2, 3), (3, 2), (4, 1)$ for a total of 4 pairs of positive integers (x, y) which would be solutions.

20. Let f be a function whose domain is all real numbers and whose function values are real numbers. Consider the following statement: "For every positive real number ϵ there is a positive real number δ so that, for each real number x it is true that if $0 < |x-2| < \delta$ then $|f(x)-5| < \epsilon$." Which of the following statements about f is true if and only if the statement above is false?

    A. "For every positive real number ϵ there is some positive real number δ so that for some real number x it is true that $0 < |x-2| < \delta$ but $|f(x)-5| \geq \epsilon$."

    B. "For some positive real number ϵ there is a positive real number δ so that for some real number x it is true that $0 < |x-2| < \delta$ and $|f(x)-5| \geq \epsilon$."

    C. "For every positive real number δ there is no positive real number ϵ so that $|x-2| \geq \epsilon$ or $|x-2| = 0$ and $|f(x)-5| < \delta$."

    D. "For each negative real number ϵ there is no negative real number δ so that $\delta < |x-2|$ and $\epsilon < |f(x)-5|$."

    E. "There is some positive real number ϵ so that for each positive real number δ there is some real number x so that $0 < |x-2| < \delta$ and $|f(x)-5| \geq \epsilon$."

Solution: It may be helpful to use rules for logical negation with symbols. If $P(x)$ is the statement $0 < |x-2| < \delta$ and $Q(x)$ is the statement $|f(x)-5| < \epsilon$ then we could formulate the statement as follows: $\forall\epsilon > 0(\exists\delta > 0(\forall x(\neg P(x) \vee Q(x))))$, the negation of which switches the "for every" symbol to "there exists" and vice versa, "or" to "and," and negates atomic statements about free variables to give $\exists\epsilon > 0(\forall\delta > 0(\exists x(P(x) \wedge \neg Q(x))))$, which is the same as E.

21. Let $\sigma(n)$ be the sum of the positive divisors of n (including 1 and n itself) for each positive integer n. Find the largest of the numbers $\sigma(1), \sigma(2), ..., \sigma(50)$.

    A. 106

    B. 88

    C. 93

    D. 122

    E. 124 (not 96 as erroneously listed on the exam)

Solution: Note: This problem did not have the correct answer choice listed and was an error on the exam. The prime numbers less than 50 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47. All divisors of numbers less than 50 must be products of such prime numbers. The positive divisors of each integer are one and the number itself, each power of each of its prime factors that divides the integer and each product of the powers of prime factors dividing the integer in its prime factor decomposition. For each prime number p, then, $\sigma(p) = p+1$. None of the prime numbers listed could be numbers at which σ achieves a maximum since $\sigma(50) = 1+2+5+10+25+50 = 93$ exceeds $\sigma(p)$ for each prime $p < 50$.

Likewise, a product of two prime numbers to the first power p, q has $\sigma(pq) = p+q+1+pq$. For an integer less than fifty to be of this form p and q would both have to be less than 25, so the sum would be less than 93 again. Thus, we are only interested in products of powers of two primes raised to powers where at least one power is higher than one, or products of three or more prime numbers, and we would, of course, want the power of each prime in the decomposition to be as high as possible without the product exceeding fifty.

This narrows down the list to $(2^2)(3^2)$, $(2^4)(3)$, $(2)(5^2)$, $(2^2)(7)$, $(2^2)(11)$, $(3^2)(5)$, $(2)(3)(7)$, $(2)(3)(5)$ as the only possibilities. Checking each directly, we have $\sigma(36) = 1+2+3+4+6+9+12+18+36 = 91$, $\sigma(42) = 1+2+3+6+7+14+21+42 = 96$, $\sigma(28) = 1+2+4+7+14+28 = 56$, $\sigma(44) = 1+2+4+11+22+44 = 84$, $\sigma(45) = 1+3+9+15+45 = 73$, $\sigma(30) = 1+2+5+6+10+15+30 = 69$ and $\sigma(48) = 1+2+3+4+6+8+12+16+24+48 = 124$. The largest of these is $\sigma(48) = 124$ (on the exam the choice listed was 96, which was incorrect, and grading adjustments to the exam results had to be made accordingly).

22. What is the ones digit of $2^{2024} + 3^{2024} + 4^{2024} + 5^{2024} + 6^{2024}$?

    A. 2
    B. 4
    C. 6
    D. 8
    E. 0

Solution: We use modular arithmetic mod ten. First, note that $3^2 = 9$ which is congruent to -1 mod 10. Thus, $(3^2)^{1012}$ is congruent to $(-1)^{512} = 1$, so the ones digit of 3^{2024} is 1. Next, notice that every power of five has a ones digit of 5, so the ones digit of 5^{2024} is 5. We next observe that $2^3 = 8$ which is -2 mod ten, so $2^{2024} = 8^{253}$ would have the same ones digit as $(-2)^{253} = ((-2)^{11})^{23}$. Raising $(-2)^{11} = -2048$ which is has a ones digit congruent to 2, so the ones digit of $(-2)^{253}$ is the same as the ones digit of 2^{23}. Also, 2^{11} has a ones digit congruent to -2, which means that $2^{23} = (2^{11})(2^{11})(2)$ has a ones digit congruent to $2((-2)(-2))$ which means the ones digit is 8. Since 4^{2024} is the square of 2^{2024}, the ones digit is the same as that of 64, namely 4, for 4^{2024}. Any power of six ends in a six, so raising 6^{2024} ends in a ones digit of 6. Adding those remainders we have $8+1+4+5+6 = 24$, which is the same as 4 mod 10. Hence, the ones digit is a 4.

23. Evaluate the series $\sum_{i=1}^{\infty} \frac{2i-1}{2^i} = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \frac{9}{32} + ...$

    A. Infinity
    B. 4
    C. 2
    D. $\frac{19}{18}$

E. 3

Solution: We split the series into separate geometric series, writing the sum as:
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 1$
$+$
$\frac{2}{4} + \frac{2}{8} + \frac{2}{16} + ... = 1$
$+$
$\frac{2}{8} + \frac{2}{16} + \frac{2}{32} + ... = \frac{1}{2}$
$+$
$\frac{2}{16} + \frac{2}{32} + \frac{2}{64} + ... = \frac{1}{4}$
$+$
...
The totals of these series starting at the second series sum form the geometric series $1 + \frac{1}{2} + \frac{1}{4} + ... = 2$. Thus, the total (with the first series sum) is 3.

24. A car travels along a straight road. The velocity of the car is $v(t) = 3t^2 - 18t + 24$ meters per second, where t is in seconds. Find the total distance traveled by the car in the time interval from $t = 0$ to $t = 4$ seconds.

  A. 16 meters

  B. 15 meters

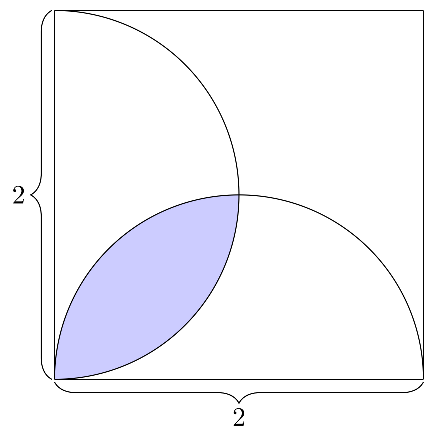
  C. 24 meters

  D. 80 meters

  E. 32 meters

Solution: The car is moving forward when the velocity is positive and backwards when the velocity is negative. To find the distance traveled, we must take the sum of the absolute values of the displacements over each maximal interval on which the velocity does not change sign. Setting $3t^2 - 18t + 24 = 0$ we have $3(t^2 - 6t + 8) = 0$, so $3(t-4)(t-2) = 0$, so the zeroes of the velocity are 2 and 4. Checking the sign of the velocity on each interval tells us that the velocity is non-negative on $[0, 2]$ and non-positive on $[2, 4]$. We then take an antiderivative of velocity to get $s = t^3 - 9t^2 + 24t$ (linear position if we start at position $s = 0$ when $t = 0$). The total distance traveled is then $\left|\int_0^2 v(t)dt\right| + \left|\int_2^4 v(t)dt\right| = \left|t^3 - 9t^2 + 24t\Big|_0^2\right| + \left|t^3 - 9t^2 + 24t\Big|_2^4\right| = 20 + (20 - 16) = 24$ meters.

25. Two semicircles are inscribed in a square of side length 2 with the base of each semicircle equal to a side of the square, where the two sides meet at a vertex. Find the area of the intersection of the two circles.



A. $1 + \frac{\pi}{4}$
B. $\frac{\pi}{2} - 1$
C. $4 - \pi$
D. $\frac{3\pi - 4}{4}$
E. $\frac{\pi + 1}{4}$

Solution: Label the vertex where the bases of the two semicircles meet O, the point of intersection of the two circles P and label the point a distance one along the bottom edge of the square (the midpoint) M, and the point a distance one along the left edge H. Thus, $MO = MP = HP = HO = 1$. We can also see that $\angle MOH = \frac{\pi}{2}$ and $\angle OHP = \angle OMP$, which we deduce that each angle of quadrilateral $MOHP$ is $\frac{\pi}{2}$, so $MOHP$ is a square of side length one and therefore area 1. The $\frac{\pi}{4}$ is the area of quarter circle with vertices M, P, O and the quarter circle with vertices H, P, O, so $2(\frac{\pi}{4}) = S + 1$, where S is the area of the shaded region (which is included twice in the sum of the areas of the quarter circles). Thus result is $S = \frac{\pi}{2} - 1$.

26. A drawer contains eight blue socks, eight white socks, and eight green socks. If a person randomly draws four socks from the drawer, what is the probability that the person has drawn two socks of one color and two socks of another color (two matching pairs of socks of two different colors)?

A. $\frac{56}{231}$
B. $\frac{1}{2}$
C. $\frac{112}{251}$
D. $\frac{56}{253}$
E. $\frac{1}{4}$

Solution: The number of ways to draw a pair of white socks and a pair of green socks is $\binom{8}{2}\binom{8}{2}$. The number of ways to draw a pair of white and blue and the number of ways to draw a pair of blue and green is the same, so the number of ways to draw two pairs of different colors is $3\binom{8}{2}\binom{8}{2} = 3(28)^2$. The number of ways to draw four socks from twenty four is $\binom{24}{4} = (23)(22)(21)$. Thus, the probability of drawing two socks of two different colors is $\frac{(3)(7^2)(4^2)}{(23)(22)(21)} = \frac{56}{253}$.

27. A quadrilateral $ABCD$ is inscribed in a circle. The side lengths of the quadrilateral are $AB = 1$, $BC = 7$, $CD = 5$ and $DA = 5$. Find the area enclosed by the quadrilateral.
    - A. $2\sqrt{165}$
    - B. 12
    - **C.** 16
    - D. 35
    - E. There is not enough information to uniquely determine the area

Solution: First, note that by the Inscribed Quadrilateral Theorem opposite angles of a quadrilateral inscribed inside a circle are supplementary, and therefore the cosines of opposite angles are negatives of each other. Let $l = AC$, and let $\theta = \angle ABC$. Then $\pi - \theta = \angle ADC$. By the law of cosines $\cos(\theta) = \frac{7^2 + 1^2 - l^2}{2(1)(7)} = -\cos(\pi - \theta) = -\frac{5^2 + 5^2 - l^2}{(2)(5)(5)}$. This equation is true if the numerators are both zero, which happens if $l^2 = 50$, which means that $\cos(\theta) = \cos(\pi - \theta) = 0$, so $\theta = \frac{\pi}{2}$. Thus, $\triangle ABC$ and $\triangle ADC$ are right triangles, so to find their areas we just multiply their side lengths (the sides that are not l) and divide by two. The total area enclosed by quadrilateral $ABCD$ is therefore $\frac{7}{2} + \frac{25}{2} = 16$.

28. A total of 200 students are enrolled in the performing arts program at Heresville High School. The program has three classes: orchestra, choir and theater. There are 75 students in choir, 91 students in orchestra and 119 students in theater. There are 26 students enrolled in both choir and orchestra, 39 students in both orchestra and theater and 34 students in both theater and choir. How many students are enrolled in all three classes?
    - A. 7
    - **B. 14**
    - C. 20
    - D. 42
    - E. 66

Solution: Let C, O, T denote the sets of students enrolled in choir, orchestra, and theater respectively. Then

$$|C \cup O \cup T| = |C| + |T| + |O| - |C \cap T| - |C \cap O| - |O \cap T| + |C \cap O \cap T|.$$

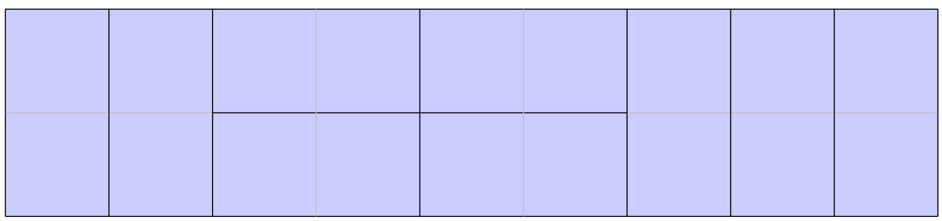
Substituting the known cardinalities, we have

$$200 = 75 + 119 + 91 - (26 + 39 + 34) + |C \cap O \cap T|,$$

so $|C \cap O \cap T| = 14$. This can also be solved using systems of equations.

29. How many ways can you cover a rectangular grid consisting of two squares by nine squares with domino-shaped tiles which can each tile two adjacent squares horizontally or two adjacent squares vertically?

Sample Domino Tiling of 2 by 9 Grid



A. 512
B. 60
C. 48
D. 55
E. 62

Solution: Let S_n be the number of ways to tile a $2 \times n$ rectangular grid with domino shaped tiles. Then $S_1 = 1$ and $S_2 = 2$. For $n > 2$, we can tile the upper left square with a vertical domino, in which case there are S_{n-1} ways to tile the rest of the grid, or we can tile the upper left square with a horizontal domino in which case the lower left square must be tiled with a horizontal domino, leaving S_{n-2} ways to tile the rest of the grid. This means that $S_n = S_{n-1} + S_{n-2}$ for $n > 2$. Hence, the number of ways to tile a $2 \times n$ grid is is the $n + 1$st member of the Fibonacci sequence. The members of this sequence are $1, 2, 3, 5, 8, 13, 21, 34, 55$ and so on, with $S_9 = 55$.

30. Twenty five marbles are placed into two bags (so the total number of marbles in the two bags is twenty five, not the number in each individual bag). The marbles are all either black or white. With this distribution of marbles, if a marble is chosen at random from each bag then the probability that two black marbles are chosen is $\frac{17}{50}$. Let $\frac{m}{n}$ be the probability of picking two white marbles if you choose one marble at random from each bag, where m and n are relatively prime positive integers. What is $m + n$?

   A. 13

   B. 221

   C. 27

   D. 109

   E. 5

Solution: Let b_1, b_2 be the number of black marbles in each bag. Then $\left(\frac{b_1}{t_1}\right)\left(\frac{b_2}{t_2}\right) = \frac{17}{50}$, where we assume $t_1 > t_2$ are the totals of the marbles in the bags with b_1 and b_2 black marbles respectively. It follows that the total number of marbles in each bag must be divisible by five (or $t_1 t_2$ in reduced terms could not be divisible by five). Thus, the possible totals are $(15, 10)$, $(20, 5)$ if we list the larger number first. If the totals are $(20, 5)$ then $t_1 t_2 = 100$, so we know that $b_1 b_2 = 34$, which could only happen if $b_1 = 17$ and $b_2 = 2$. If the total numbers of marbles are $(15, 10)$ then $t_1 t_2 = 150$, so $b_1 b_2 = 51$. There is no case where this is possible, however, since one of the factors of the numerator must be divisible by 17 for this to happen, and neither bag contains seventeen or more marbles.

Hence, the only possibility is that there are 20 marbles in the first bag of which 3 are white and 5 in the second, of which 3 are white. Hence, the probability of drawing a white marble from each bag is $\left(\frac{3}{20}\right)\left(\frac{3}{5}\right) = \frac{9}{100}$, so $m + n = 109$.