State Math Contest 2024 Senior Level

Instructions:

- Calculators, cell phones and other computational devices are not permitted (you can only use pens, pencils and paper to work on your answers, and then mark your answers with a number two pencil on the answer sheet).
- Correct answers are worth 5 points. Unanswered questions will be given 1 point. Incorrect answers will be worth 0 points. *This means that it will not, on average, increase your score to guess answers randomly.*
- Fill in the answers on the answer sheet using a number two pencil.
- Time limit: 120 minutes.
- When you are finished, please give the exam and any scratch paper to the test administrator.
- Good luck!

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1. Solve $\log x + \log(x + 15) = 2$.

   A. $x = 5$

   B. $x = 5$ or $x = 20$

   C. $x = 5$ or $x = -20$

   D. $x = 15$

   E. $x = 10$

2. Given that $i^2 = -1$, find $(1 + i\sqrt{3})^{10}$.

   A. $-512 - 512\sqrt{3}i$

   B. $-256 + 265\sqrt{3}i$

   C. $-512\sqrt{3} - 512i$

   D. $-128 + 128\sqrt{6}i$

   E. $1024 + 1024\sqrt{3}i$

3. Let $f(x - 1) + f(x + 1) = f(x)$ for all integers x. If $f(0) = 2$ and $f(1) = 4$ then find $f(2022) + f(2023) + f(2024)$.

   A. 12

   B. 8

   C. 337

   D. 0

   E. 4

4. Evaluate the sum $\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + ... + \frac{1}{(19)(20)}$.

   A. $\frac{19}{20}$

   B. $\frac{19!-1}{20!}$

   C. $\frac{121}{20!}$

   D. 0.96

   E. 1.04

5. Let C be the circle of radius 24 centered at the origin in the plane. Let R be a radius of C (a line segment from the origin to a point of C). Let P be the perpendicular bisector of R, which separates the plane into sides U and V, where U contains the origin. Find the length of the arc A which consists of the points of C which are not in U.

   A. 8π

   B. 40

   C. 0

   D. 6π

   E. 16π

6. How many integers between 1 and 10,000 are divisible by each of the numbers $2, 3, 4, 5, 6, 7, 8, 9$?

   A. 0

   B. 1

   C. 3

   D. 5

   E. 28

7. Let $x^2 + y^2 - 7x + 4y + 13 = 0$. What is the smallest possible value of $3x - 2y$?

   A. 4

   B. $2\sqrt{3}$

   C. -4

   D. 8

   E. There is no smallest value

8. A survey team places a 24-foot tall vertical pole on top of a (rather short) hill. Someone standing some unknown distance horizontally from the hill measures the angle of elevation to the bottom of the pole to be $30°$ and the angle of elevation to the top of the pole to be $45°$. Find the height of the hill.

   A. 48 feet

   B. $48(\sqrt{2}-1)$ feet

   C. $\dfrac{24(\sqrt{3}-1)}{5}$ feet

   D. $12(\sqrt{3}+1)$ feet

   E. 24 feet

9. How many different "words" can be made out of the letters of SASSAFRAS, if a "word" means any string of all nine letters? For example, FRAAASSSS would be a word.

   A. 62

   B. 2048

   C. 2520

   D. 362880

   E. 15120

10. Let $\sin(x) - \cos(x) = \dfrac{1}{3}$. Find $\sin(2x)$.

    A. $\dfrac{5}{9}$

    B. $\dfrac{8}{9}$

    C. $\dfrac{7}{9}$

    D. $\dfrac{2}{9}$

    E. $\dfrac{4}{9}$

11. Let r and a be positive real numbers so that a disk of radius r is small enough to fit inside an equilateral triangle T of side length a. Find the area of the region enclosed by T consisting of all points which are not enclosed by any circle of radius r which is enclosed by T.

    A. $(a^2 - r^2)(3\sqrt{3} - \pi)$

    B. $\dfrac{a^2}{r^2}(6\sqrt{2} - 2\pi)$

    C. $r^2(\dfrac{\sqrt{3}}{4} - \dfrac{\pi}{12})$

    D. $r^2(3\sqrt{3} - \pi)$

    E. $r^2(\pi - \sqrt{3})$

12. Let a, b, c, x, y, z be non-zero complex numbers so that $a = \dfrac{b+c}{x+1}$, $b = \dfrac{c+a}{y+1}$, $c = \dfrac{a+b}{z+1}$. If $xy + yz + zx = 100$ and $x + y + z = 50$, find the value of xyz.

A. -96

B. -94

C. -82

D. -86

E. -92

13. Which of the following is the graph of $x^2 + 2\sqrt{3}xy + 3y^2 + 6x + 16 = 0$?

A. A parabola

B. An ellipse

C. A hyperbola

D. A line

E. None of these

14. Let S consist of all numbers which can be obtained by starting at 0 and then applying the functions $f(x) = x + 1$ and $g(x) = \dfrac{x}{2}$ finitely many times (in any possible function composition order; for example, $g(f(f(g(g(0)))))$ would be one of the points of S).

Which of the following is true?

(I) If n is in S then n^2 is in S.
(II) If n is in S then $n - 1$ is in S.
(III) If n is in S then $n^2 - n$ is in S.

A. (I) only

B. (I) and (II) only

C. (I), (II), and (III)

D. (II) and (III) only

E. None of these are true

15. Let $f(x) = e^{3x+1}$. Find $\displaystyle\lim_{x\to 0} \frac{f(f(x)) - f(e)}{x}$.

A. $3e^{e^e}$

B. $\dfrac{1}{e}$

C. $9e^{3e+2}$

D. $9e^{6e^2}$

E. 0

16. Find the smallest natural number m so that $\sum_{n=1}^{m} \lfloor \log_{10}(n) \rfloor > 2024$. (The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x).

    A. 10^{2024}

    B. 1035

    C. 899

    D. 1190

    E. 1044

17. Let $f(x) = ax^2 + bx + c$ be a quadratic function so that $f(1) + f(2) + ... + f(n) = n^3$ for every positive integer n. What is abc?

    A. -6

    B. -8

    C. -9

    D. -10

    E. -13

18. Let $f(x)$ be a degree four polynomial so that $f(x) < 0$ for all real x. Which of the following must be true about $g(x) = f(x) + f'(x) + f''(x) + f'''(x) + f^{(4)}(x)$?

    A. $g'''(x) = 0$ for all x.

    B. $g(x)$ has degree five.

    C. $g(x) > 0$ for some real x

    D. $g(x) < 0$ for all real x

    E. None of these need to be true.

19. What is the number of positive integer solutions (x, y) of the equation $x^2 + y^2 = 4x + 4y + 5 - 2xy$?

    A. 0

    B. 1

    C. 3

    D. 4

    E. Infinitely many

20. Let f be a function whose domain is all real numbers and whose function values are real numbers. Consider the following statement: "For every positive real number ϵ there is a positive real number δ so that, for each real number x it is true that if $0 < |x-2| < \delta$ then $|f(x)-5| < \epsilon$." Which of the following statements about f is true if and only if the statement above is false?

   A. "For every positive real number ϵ there is some positive real number δ so that for some real number x it is true that $0 < |x-2| < \delta$ but $|f(x)-5| \geq \epsilon$."

   B. "For some positive real number ϵ there is a positive real number δ so that for some real number x it is true that $0 < |x-2| < \delta$ and $|f(x)-5| \geq \epsilon$."

   C. "For every positive real number δ there is no positive real number ϵ so that $|x-2| \geq \epsilon$ or $|x-2| = 0$ and $|f(x)-5| < \delta$."

   D. "For each negative real number ϵ there is no negative real number δ so that $\delta < |x-2|$ and $\epsilon < |f(x)-5|$."

   E. "There is some positive real number ϵ so that for each positive real number δ there is some real number x so that $0 < |x-2| < \delta$ and $|f(x)-5| \geq \epsilon$."

21. Let $\sigma(n)$ be the sum of the positive divisors of n (including 1 and n itself) for each positive integer n. Find the largest of the numbers $\sigma(1), \sigma(2), ..., \sigma(50)$.

   A. 106

   B. 88

   C. 93

   D. 122

   E. 96 (this was an error and should have been 124)

22. What is the ones digit of $2^{2024} + 3^{2024} + 4^{2024} + 5^{2024} + 6^{2024}$?

   A. 2

   B. 4

   C. 6

   D. 8

   E. 0

23. Evaluate the series $\sum_{i=1}^{\infty} \frac{2i-1}{2^i} = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \frac{9}{32} + ...$

   A. Infinity

   B. 4

   C. 2

   D. $\frac{19}{18}$

   E. 3

24. A car travels along a straight road. The velocity of the car is $v(t) = 3t^2 - 18t + 24$ meters per second, where t is in seconds. Find the total distance traveled by the car in the time interval from $t = 0$ to $t = 4$ seconds.

    A. 16 meters

    B. 15 meters

    C. 24 meters

    D. 80 meters

    E. 32 meters

25. Two semicircles are inscribed in a square of side length 2 with the base of each semicircle equal to a side of the square, where the two sides meet at a vertex. Find the area of the intersection of the two circles.

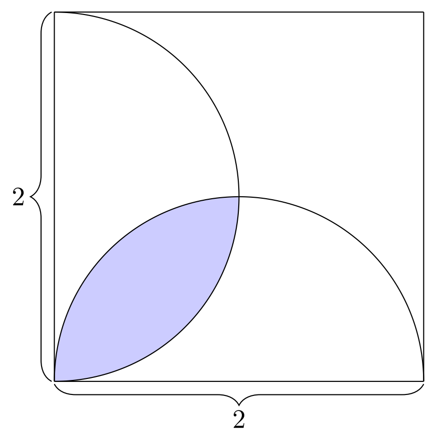
    A. $1 + \frac{\pi}{4}$

    B. $\frac{\pi}{2} - 1$

    C. $4 - \pi$

    D. $\frac{3\pi - 4}{4}$

    E. $\frac{\pi + 1}{4}$



26. A drawer contains eight blue socks, eight white socks, and eight green socks. If a person randomly draws four socks from the drawer, what is the probability that the person has drawn two socks of one color and two socks of another color (two matching pairs of socks of two different colors)?

    A. $\frac{56}{231}$

    B. $\frac{1}{2}$

    C. $\frac{112}{251}$

    D. $\frac{56}{253}$

    E. $\frac{1}{4}$

27. A quadrilateral $ABCD$ is inscribed in a circle. The side lengths of the quadrilateral are $AB = 1$, $BC = 7$, $CD = 5$ and $DA = 5$. Find the area enclosed by the quadrilateral.

    A. $2\sqrt{165}$

    B. 12

    C. 16

    D. 35

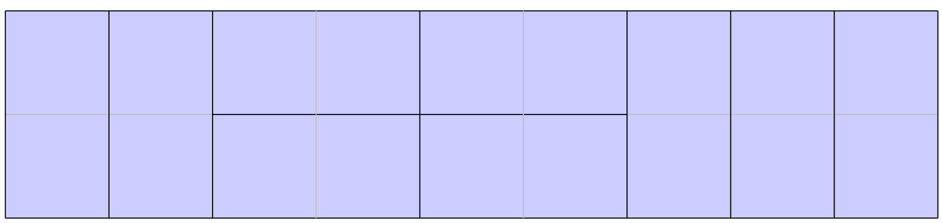
    E. There is not enough information to uniquely determine the area

28. A total of 200 students are enrolled in the performing arts program at Heresville High School. The program has three classes: orchestra, choir and theater. There are 75 students in choir, 91 students in orchestra and 119 students in theater. There are 26 students enrolled in both choir and orchestra, 39 students in both orchestra and theater and 34 students in both theater and choir. How many students are enrolled in all three classes?

    A. 7
    B. 14
    C. 20
    D. 42
    E. 66

29. How many ways can you cover a rectangular grid consisting of two squares by nine squares with domino-shaped tiles which can each tile two adjacent squares horizontally or two adjacent squares vertically?

Sample Domino Tiling of 2 by 9 Grid



    A. 512
    B. 60
    C. 48
    D. 55
    E. 62

30. Twenty five marbles are placed into two bags (so the total number of marbles in the two bags is twenty five, not the number in each individual bag). The marbles are all either black or white. With this distribution of marbles, if a marble is chosen at random from each bag then the probability that two black marbles are chosen is $\frac{17}{50}$. Let $\frac{m}{n}$ be the probability of picking two white marbles if you choose one marble at random from each bag, where m and n are relatively prime positive integers. What is $m+n$?

    A. 13
    B. 221
    C. 27
    D. 109
    E. 5