State Math Contest 2024 Junior Level Solutions

Instructions:

- Calculators, cell phones and other computational devices are not permitted (you can only use pens, pencils and paper to work on your answers, and then mark your answers with a number two pencil on the answer sheet).
- Correct answers are worth 5 points. Unanswered questions will be given 1 point. Incorrect answers will be worth 0 points. *This means that it will not, on average, increase your score to guess answers randomly.*
- Fill in the answers on the answer sheet using a number two pencil.
- Time limit: 120 minutes.
- When you are finished, please give the exam and any scratch paper to the test administrator.
- Good luck!

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1. Andrew sold a lamp for 5 dollars less than the listed price and received 10% of the sale revenue as a commission. Brittany sold the same type of lamp (with the same listed price) for 20 dollars less than the listed price, but she received 20% of the sale revenue as a commission. Assuming there was no sales tax, if both Andrew and Brittany received the same commission then what was the listed price of the lamp in dollars?

   A. 28
   B. 32
   C. 35
   D. 40
   E. 45

   Solution: Let x be the price of the lamp. Then $\frac{10}{100}(x-5) = \frac{20}{100}(x-20)$, so $x-5 = 2x-40$, and thus $x = 35$.

2. A polynomial of degree 4 whose coefficients are real numbers has zeros $2, 3$ and $1-i$. If the coefficient of x^4 is 1, what is the coefficient of x^3?

   A. -22
   B. -5
   C. -7
   D. 18
   E. 12

   Solution: Since the coefficients are real, the complex zeroes come in conjugate pairs, so the polynomial has form $C(x-2)(x-3)(x-(1-i))(x-(1+i))$. Since the leading coefficient is one, $C = 1$, so this becomes $(x^2-5x+6)(x^2-2x+2) = x^4 - 7x^3 + 18x^2 - 22x + 12$.

3. Let f be a function so that $f(x+y) = f(x)f(y)$ for all integers x and y. If $f(1) = 2$ then find $f(0) + f(1) + f(2) + .. + f(9)$.
   - A. 1033
   - **B. 1023**
   - C. 1212
   - D. 1022
   - E. 1134

> **Solution:** If $x = 0$ and $y = 1$ we have $f(x)f(y) = f(0+1) = 2$, so $f(0) = 1$. If we know $f(k)$ for a given positive integer k then $f(k+1) = f(k)f(1) = 2f(k)$ which means that $f(n) = 2^n$ for every integer n. Hence, $f(0) + f(1) + f(2) + .. + f(9) = \sum_{i=0}^{9} 2^i = \dfrac{1(1-2^{10})}{1-2} = 1023$.

4. A total of 50 athletes are competing in a track and field contest, each competing in at least one of only three available events. One is the fifty meter dash, one is the hammer throw, and the last is the pole vault. There are 31 competitors in the fifty meter dash, 24 in the hammer throw and 23 in the pole vaulting competition. There are 13 in both the fifty meter dash and the hammer throw, 11 in both the fifty meter dash and the pole vault, and 9 in both the pole vault and the hammer throw. How many athletes are competing in all three events?
   - A. 0
   - B. 3
   - C. 4
   - D. 7
   - **E. 5**

> **Solution:** Let F, H, P denote the sets of athletes competing in fifty meter dash, hammer throw, and pole vaulting respectively. Then
>
> $$|F \cup H \cup P| = |F| + |H| + |P| - |F \cap H| - |H \cap P| - |F \cap P| + |F \cap H \cap P|.$$
>
> Substituting the known cardinalities, we have
>
> $$50 = 31 + 24 + 23 - (13 + 11 + 9) + |F \cap H \cap P|,$$
>
> so $|F \cap H \cap P| = 5$.

5. If x^{2024} is divided by $x - 2$ then what is the remainder?
   - A. 0
   - B. 1
   - C. 16320
   - **D.** 2^{2024}
   - E. x

> **Solution:** Using the Remainder Theorem, the remainder when a polynomial $P(x)$ is divided by $x - c$ is always $P(c)$. In this case, $P(2) = 2^{2024}$.

6. What is the sum of all positive integers less than 1000 that are divisible by 7 but not by 5?

   A. 70771

   B. 71071

   C. 70071

   D. 56861

   E. 92150

> **Solution:** The positive integers divisible by seven less than 1000 form a finite arithmetic sequence whose sum is $7+14+21+28+\cdots+994 = 7(1+2+3+\cdots+142) = 7(143)(71)$. We then have to subtract from this sum the sum of all of the numbers which are also divisible by five, which is $7(5+10+15+\cdots+140) = 7(5)(1+2+\cdots+28) = 7(5)(14)(29)$. The difference is $7(10153-2030) = 71071-14210 = 56861$.

7. Let $f(x+y) = f(x)+f(y)+2xy-2$ for all real numbers x, and let $f(1) = 6$. Find $f(5)$.

   A. 25

   B. 13

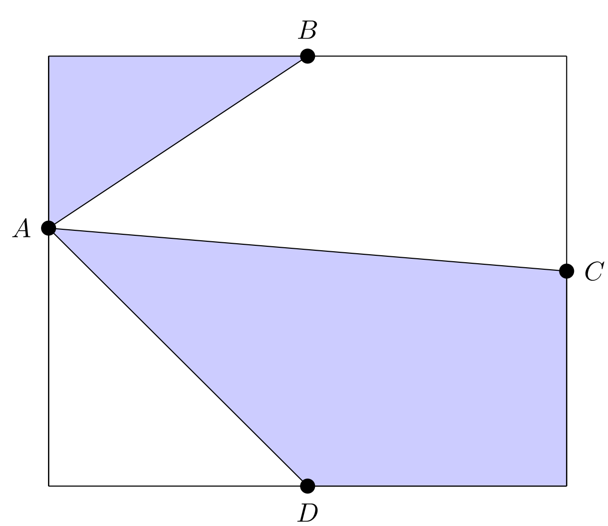
   C. 48

   D. 56

   E. 42

> **Solution:** We have $6 = f(0+1) = f(0)+f(1)-2$, so $f(0) = 2$. Also, $f(1+1) = f(1)+f(1) = 12$ and $f(3) = f(2+1) = f(2)+f(1)+2 = 20$, and $f(4) = f(3+1) = f(3)+f(1)+4 = 30$ and $f(5) = f(4+1) = f(4)+f(1)+6 = 42$.
>
> Note that there is such a function, $f(x+y) = (x+1)(x+2)$, since $f(x)+f(y)+2xy-2 = x^2+3x+2+y^2+3y+2+2xy-2 = x^2+3x+3y+2xy+2 = ((x+y)+1)((x+y)+2)$.

8. The rectangle shown has an area of 36. B, C, D are the midpoints of the sides of the rectangle shown. The point A is three fifths of the distance from the bottom of the side shown. Find the area of the shaded region.



   A. 9

   B. 24

C. 18

D. 12

E. 27

Solution: First name half of the width of the rectangle w and name half of the height of the rectangle l. Let P be the lower right vertex of the rectangle and let Q be the vertex at the upper left-hand corner of the triangle. Let h be the height of A from the base of the rectangle (so $2l - h$ is the distance from the top). Then the area enclosed by $\triangle ACP$ is lw, the area enclosed by triangle $\triangle APD$ is $\frac{wh}{2}$, and the area enclosed by triangle $\triangle ABQ$ is $(2l-h)\frac{w}{2}$. Adding these together we get that the area of the shaded region is $2lw$, which is half of $4lw$, the area of the rectangle. Since this rectangle has area 36, the area of the shaded region is 18. Notice that this solution would have worked the same way no matter where A was positioned along the left side of the rectangle.

9. Out of a class of ten people, three people must be chosen to participate in an athletic event and a fourth person must be chosen to go along to help with the equipment and registration as a team manager. How many ways can a manager and a set of three participants be selected from the class of ten people?

A. 240

B. 1200

C. 840

D. 210

E. 720

Solution: There are ten ways to choose the manager, and then nine choose three ways to choose the set of participants, for a total of $\frac{(10)(9!)}{(6!)(3!)} = 840$ ways to choose the manager and the participants. We could think of it as a permutation problem choosing four individuals with three identical from a set of ten.

10. If $6x + 5y = 4$, find $64^x 32^y$.

A. 16

B. 32

C. 4

D. 64

E. 128

Solution: $64^x 32^y = 2^{6x} 2^{5y} = 2^{6x+5y} = 2^4 = 16$.

11. Find the units digit of $7^{2031} \times 3^{1202} \times 6^{999}$.

   A. 0
   B. 1
   C. 2
   D. 7
   E. 8

> **Solution:** We use modular arithmetic. First, note that raising six to any positive integer power has a ones digit of 6 (which is congruent to -4 mod ten, which is slightly easier to multiply by). Also, $3^3 = 9$ which is congruent to -1 mod ten. Hence, 3^{1202} has the same ones digit as $(-1)^{1101}$ which is the same as -1 mod ten (giving a ones digit of 9, but -1 is simpler to multiply by). Next, note that $7^2 = 49$ which is congruent to -1 mod ten, so 7^{2030} is congruent to $(-1)^{1015} = -1$ mod ten. Multiplying by another 7 we get that 7^{2031} is congruent to -7 mod ten, so the ones digit is 3. Multiplying $(3)(-1)(-4) = 12$ which is congruent to 2 mod ten, which means that the units digit is 2.

12. A pump can fill a swimming pool in 8 hours. A second pump can fill the same pool in 10 hours. When the pool's drain is open, the (completely filled) pool drains in 20 hours. Assume both pumps are operating and the drain is open, and each pump adds water at a constant rate and water is also removed through the drain at a constant rate. How long will it take to fill the pool?

   A. $5\ \frac{4}{7}$ hours
   B. $5\ \frac{2}{7}$ hours
   C. $5\ \frac{3}{7}$ hours
   D. $5\ \frac{1}{7}$ hours
   E. $5\ \frac{5}{7}$ hours

> **Solution:** The rate of water pumped into the pool in pools per hour for the first pump is $\frac{1}{8}$, for the second pump is $\frac{1}{10}$ and for the drain is $-\frac{1}{20}$, so the total "pools per hour" rate of filling the pool if both pumps and the drain are active is $\frac{1}{10} + \frac{1}{8} - \frac{1}{20} = \frac{7}{40}$ pools per hour. Thus, the number of hours to fill the pool is $\frac{40}{7} = 5 + \frac{5}{7}$ hours.

13. A fair coin is flipped three times. What is the probability that two or more heads are flipped given that one of the flips was a head?

   A. $\frac{1}{2}$

   B. $\frac{2}{3}$

   C. $\frac{1}{3}$

   D. $\frac{5}{7}$

   E. $\frac{4}{7}$

> **Solution:** Given that we know at least one head has been flipped, the possible outcomes are $HHH, HHT, HTH, THH, HTT, THT, TTH$. Of those seven equally probable outcomes, four have at least two flipped heads.

14. Let $a = \sqrt{6 + \sqrt{6 + \sqrt{6 + ...}}}$, and $b = \sqrt{12 - \sqrt{12 - \sqrt{12 - ....}}}$ Find ab.

   A. 9

   B. 6

   C. $\sqrt{48}$

   D. Infinity

   E. 8

> **Solution:** First, note that $a = \sqrt{6 + a}$, which means that $a^2 = 6 + a$ and $a^2 - a - 6 = 0$, so $(a - 3)(a + 2) = 0$ and therefore $a = 3$ (since a is positive). Likewise, $b = \sqrt{12 - b}$ which means that $b^2 = 12 - b$, so $b^2 + b - 12 = 0$ and therefore $(b - 3)(b + 4) = 0$, so $b = 3$ also. Hence $ab = 9$.

15. Which of the following numbers is the smallest?

   A. $2\sqrt[3]{10}$

   B. $\frac{3\pi}{2}$

   C. $3\sqrt{2}$

   D. $\log_5 4000$

   E. $\sqrt[6]{6000}$

> **Solution:** Taking the sixth power of $2\sqrt[3]{10}$ gives 6400, taking the sixth power of $3\sqrt{2}$ gives $729 * 8 = 5832$. Taking the sixth power of $\sqrt[6]{6000}$ gives 6000, so $3\sqrt{2}$ is the smallest of the three. Furthermore, $(1.5)^2 = 2.25 > 2$, so $3\sqrt{2} < 3 * (1.5) = 4.5 < \frac{3\pi}{2}$ since $\pi > 3$. Finally, $\log_5 4000 > \log_5 3125 = \log_5 5^5 = 5$. Thus, the smallest of these five numbers is $3\sqrt{2}$.

16. Find the statement which is logically equivalent to the statement "If either of A or B are true then it follows that it is not true that both C and D are false."

    A. C is true or A is false

    B. If C is true then A is false.

    C. Either C and D are both true or A and B are both false.

    D. Either it is true that C is true or D is true, or it is true that A is false and B is false.

    E. Either it is true that C is true or D is true, or it is true that A is false or B is false.

Solution: The implication $P \to Q$ is equivalent to $\neg P \vee Q$ logically. So, we can write the original statement as $\neg(A \vee B) \vee \neg(\neg C \wedge \neg D)$. Since $\neg(A \vee B)$ is equivalent to $\neg A \wedge \neg B$ and $\neg(\neg C \wedge \neg D)$ is equivalent to $C \vee D$, this is the same as D.

17. Which of the following is equivalent to $\sin(2\tan^{-1}(x))$

    A. $\dfrac{x}{x^2+1}$

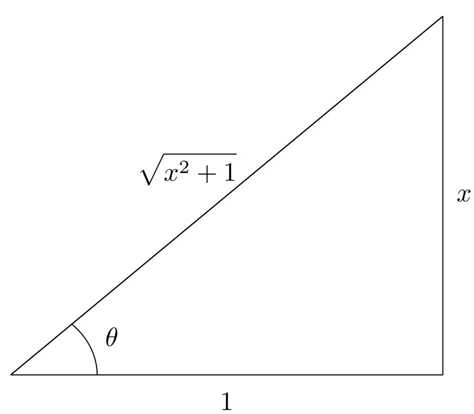
    B. $\dfrac{2x}{x^2+1}$

    C. $\dfrac{2x}{\sqrt{x^2+1}}$

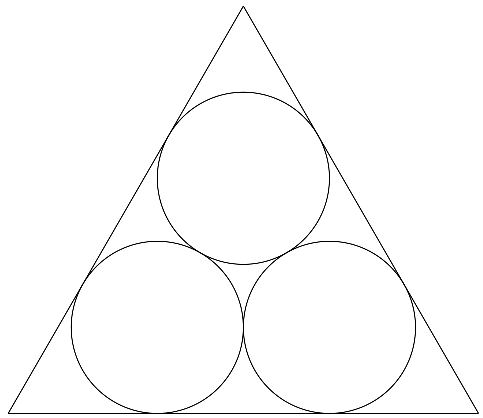
    D. $\dfrac{x^2}{x^2+1}$

    E. $\dfrac{x}{\sqrt{x^2+1}}$.

Solution: First, observe that $\sin(2\tan^{-1}(x)) = 2\sin(\tan^{-1}(x))\cos(\tan^{-1}(x))$. Then, we look at the triangle below with the angle θ whose tangent is x and see $\sin(\theta) = \dfrac{x}{\sqrt{1+x^2}}$ and $\cos(\theta) = \dfrac{1}{\sqrt{1+x^2}}$. Plugging in yields $\dfrac{2x}{x^2+1}$.



18. Three disks of radius one are mutually externally tangent to each other. Find the area enclosed by the circumscribed equilateral triangle containing the three disks.



A. 9

B. $4 + \frac{3\sqrt{3}}{2}$

C. $\frac{9}{2}$

D. $4\sqrt{3} + 3$

E. $6 + 4\sqrt{3}$

Solution: Let A, B, and C denote the centers of the three disks starting with A at the lower left and proceeding counterclockwise. Let D, E, and F be the vertices of the triangle starting with D at the lower left and proceeding counterclockwise. Let M and N be the points at which the lower two circles intersect the bottom edge of the triangle with M on the left. We note that $MN = 2$ because the radii of each circle is one. Angle $\angle ADM = \frac{\pi}{6}$ which means that length $DM = \cot\frac{\pi}{6} = \sqrt{3}$. Hence, the length of a side of the equilateral triangle is $2 + 2\sqrt{3}$. This makes the area $\frac{\sqrt{3}}{4}(2 + 2\sqrt{3})^2 = 6 + 4\sqrt{3}$.

19. Find $(\log_2 5)(\log_5 7)(\log_7 16)$.

A. $\log_{70} 560$

B. 5

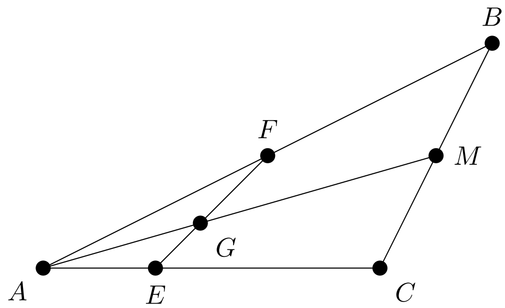
C. 4

D. 8

E. $\log_2 560$

Solution: Using the change of base formula we obtain $\frac{\log_2 5}{\log_2 2}\frac{\log_2 7}{\log_2 5}\frac{\log_2 16}{\log_2 7} = \log_2 16 = 4$.

20. A triangle $\triangle ABC$ has side lengths $AC = 27$ and $AB = 36$. Let M be the midpoint of line segment $\overline{BC}$. Let F be a point on line segment $\overline{AB}$ and E be a point on line segment $\overline{AC}$ so that $AF = 3AE$. Let G be the point of intersection of the line segments $\overline{EF}$ and $\overline{AM}$. Find $\frac{FG}{GE}$.



A. 2

B. $\frac{3}{2}$

C. $\frac{9}{4}$

D. $\frac{4}{3}$

E. Cannot be uniquely determined

Solution: First draw a line parallel to $\overline{EF}$ passing through B and let P be the point of intersection of that line with $\overline{AC}$. Since $\triangle AFE$ is similar to $\triangle ABP$, the ratio $3 = \frac{AF}{AE} = \frac{AB}{AP} = \frac{36}{AP}$, which means that $AP = 12$. Let $Q = \overline{AM} \cap \overline{BP}$.

Observe that $\sin(\theta) = \sin(\pi - \theta)$ for any angle θ, so $\sin(\angle AQP) = \sin(\angle AQB)$ and $\sin(\angle AMC) = \sin(\angle AMB)$, and since M is the midpoint of $\overline{CB}$, we have $MC = MB$. Next, we use the law of sines to see that $\frac{MC}{\sin(\angle MAC)} = \frac{27}{\sin(\angle AMC)}$ and $\frac{MB}{\sin(\angle MAB)} = \frac{36}{\sin(\angle AMB)} = \frac{36}{\sin(\angle AMC)}$. Hence, $\frac{\frac{MC}{\sin(\angle MAC)}}{\frac{MB}{\sin(\angle MAB)}} = \frac{\sin(\angle MAB)}{\sin(\angle MAC)} = \frac{\frac{27}{\sin(\angle AMC)}}{\frac{36}{\sin(\angle AMC)}} = \frac{3}{4}$.

Next, observe that since $\triangle AFE$ is similar to $\triangle ABP$, the ratio $\frac{BQ}{QP} = \frac{FG}{GE}$. By the law of sines, $\frac{QP}{\sin(\angle MAC)} = \frac{12}{\sin(\angle AQP)}$ and $\frac{BQ}{\sin(\angle MAB)} = \frac{36}{\sin(\angle AQB)} = \frac{36}{\sin(\angle AQP)}$. Thus, $\frac{BQ}{QP} = \left(\frac{\frac{BQ}{\sin(\angle MAB)}}{\frac{QP}{\sin(\angle MAC)}}\right)\left(\frac{\sin(\angle MAB)}{\sin(\angle MAC)}\right) = \left(\frac{\frac{36}{\sin(\angle AQP)}}{\frac{12}{\sin(\angle AQP)}}\right)\left(\frac{3}{4}\right) = \frac{9}{4}$.

21. A certain diet requires a person to eat exactly 28 grams of fiber, 11 grams of fat, and 14 grams of protein per day. Food A contains 2 grams of fiber, 1 gram of fat, and 2 grams of protein per serving, and food B contains 3 grams of fiber, 2 grams of fat, and 2 grams of protein per serving, and food C contains 4 grams of fiber, 1 gram of fat, and 1 gram of protein per serving. If a person eats only those three foods on this diet, then how many servings of food C must the person eat each day?

   A. 1
   B. 2
   C. 3
   D. 4
   E. 5

Solution: Let a, b, c be the number of servings consumed of foods A, B, C. Equating the amounts of fiber, fat, and protein gives us the system:

$$2a + 3b + 4c = 28$$
$$a + 2b + c = 11$$
$$2a + 2b + c = 14$$

To solve the system, we can add negative the second equation to the third to get $a = 3$. Substituting this into the first and second equations gives $3b + 4c = 22$ and $2b + c = 8$. Then adding negative four times the second of these new equations to the first gives us $-5b = -10$, so $b = 2$, which means that $c = 4$.

22. Anna and Bill play a game. Anna goes first and rolls a fair six-sided die. If she rolls a one or two then she wins. Otherwise, Bill goes and flips a fair coin. If he flips a head then he wins. Otherwise, Anna goes again, rolling her die and winning on a one or a two. If she does not win then Bill goes again, winning on a flip of heads with his coin and so on. What is the probability that Anna wins the game?

   A. $\frac{2}{3}$
   B. $\frac{1}{3}$
   C. $\frac{1}{2}$
   D. 0.6
   E. 0.4

Solution: The probability that Anna wins on her first roll is $\frac{1}{3}$. The probability that Anna does not win and then Bill does not win and then Anna wins on the second roll is $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{3^2}$. The probability that Anna and Bill fail to win twice and then Anna wins on her third roll is $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{3^3}$. Continuing, the probability that Anna and Bill fail to win $n-1$ times and then Anna wins on her nth roll is $\frac{1}{3^n}$. Thus, the probability that Anna wins is $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$.

23. Let x, y, z be real numbers so that $\dfrac{2024}{x+1} + \dfrac{2024}{y+1} + \dfrac{2024}{z+1} = 2025$. Find $\dfrac{x-1}{x+1} + \dfrac{y-1}{y+1} + \dfrac{z-1}{z+1}$.

A. $\dfrac{2022}{2024}$
B. $\dfrac{2021}{2024}$
C. $\dfrac{2019}{2024}$
D. $\dfrac{2020}{2024}$
E. $\dfrac{2017}{2024}$

Solution: First, note $\dfrac{1}{x+1}+\dfrac{1}{y+1}+\dfrac{1}{z+1} = \dfrac{2025}{2024}$. Next, observe that $\dfrac{x-1}{x+1} = \dfrac{x+1-2}{x+1} = 1 - \dfrac{2}{x+1}$ and likewise $\dfrac{y-1}{y+1} = 1 - \dfrac{2}{y+1}$ and $\dfrac{z-1}{z+1} = 1 - \dfrac{2}{z+1}$. Thus, we have $\dfrac{x-1}{x+1} + \dfrac{y-1}{y+1} + \dfrac{z-1}{z+1} = 3 - 2\left(\dfrac{1}{x+1} + \dfrac{1}{y+1} + \dfrac{1}{z+1}\right) = 3 - 2\left(\dfrac{2025}{2024}\right) = \dfrac{2022}{2024}$.

24. A triangle has side lengths 5, 6, and 7. Find the area enclosed by the triangle.

A. 12
B. $8\sqrt{2}$
C. $5\sqrt{3}$
D. $6\sqrt{6}$
E. 6

Solution: Using Heron's formula, since $(5+6+7)/2 = 9$, the area of the triangle is $\sqrt{9(9-5)(9-6)(9-7)} = 6\sqrt{6}$. The law of cosines would also yield a solution.

25. Let $n! = n(n-1)(n-2)\ldots(1)$ for any non-negative integer n. Find the number of consecutive zeros at the end of the number 250!.

A. 12
B. 50
C. 62
D. 125
E. 184

Solution: Since the power of 2 in the expansion of a factorial is always at least as large as the power of five, the number of zeroes, which corresponds to the largest power of ten that can be factored from the number, is the same as the power of five. The number of times five occurs as a factor of $(250)(249)(248)\ldots(1)$ can be determined by adding the power of five in each factor. Only the factors ending in a five or a zero are divisible by five, which are $250, 245, 240, \ldots, 5$ (a total of 50 numbers). Those divisible by 25 are $250, 225, 200, \ldots, 25$ (a total of 10 numbers), and the numbers divisible by 125 are just 125 and 250, for a total of two numbers. Thus, the sum of all powers of five of all factors is 50+10+2=62, so the number of zeroes at the end of 250! is 62.

26. Let n be the positive integer written with a string of 2024 consecutive 1's for the digits, so $n = 111...11$, a string of 2024 1's. Of the following, which is the largest number that divides n?

A. 11

B. 1111

C. 11111111 (eight 1's)

D. 1111111111111111 (16 1's)

E. 11111111111111111111111111111111 (32 1's)

Solution: The number $2024 = 8(253) = (2^3)(11)(23)$. If the number of 1's for a number $m = 111...11$ divides 2024 then m divides n. In particular, if m is a string of eight 1's then $n = m + (10^8)m + (10^{16})m + \cdots + (10^{2016})m = \sum_{i=0}^{252} 10^{8i}m$. Let k be an integer whose base ten representation is a string of a number j 1's that does not divide 2024. If $j > 2024$ then certainly k cannot divide n. Otherwise, subtract $10^{2024-j}k$ from n to leave an integer whose base ten representation is a string of $2024 - j$ 1's. If there are fewer than j 1's remaining in this representation then this is the remainder when n is divided by k. Otherwise, subtract $10^{2024-2j}k$ from what remains. The number of ones remaining is $2024 - 2j$. Continuing, we are eventually left with a number r whose base ten representation is a string of 1's of length more than zero and less than j, meaning that r is the remainder when n is divided by k, so k does not divide n. Since 16 and 32 do not divide 2024, the answer is C.

27. If $(8x)^{\log_2 3} - (2x)^{\log_2 9} = 0$, then what is $\log_2(x)$?

A. $\frac{1}{2}$

B. 1

C. 2

D. 4

E. Undefined

Solution: Since $(8x)^{\log_2 3} = (2x)^{\log_2 9}$, taking the natural logarithm of both sides gives us $(\log_2 3)(\ln 8x) = (\log_2 9)(\ln 2x)$. Since $9 = 3^2$ this means $(\log_2 3)(\ln 8x) = 2(\log_2 3)(\ln 2x)$, which implies that $\ln 8x = 2 \ln 2x$ and so $\ln 8 + \ln x = 2(\ln 2 + \ln x)$, which means that $3 \ln 2 + \ln x = 2 \ln 2 + 2 \ln x$, so $\ln x = \ln 2$, and hence $x = 2$. Thus, $\log_2 x = 1$.

28. Let $\binom{n}{k}$ refer to $\frac{n!}{(n-k)!k!}$ if $k \leq n$ and n and k are non-negative integers (where $n! = n(n-1)(n-2)\ldots(1)$ if $n \geq 1$ and $0! = 1$). Which of the following is equal to $\binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9}$?

A. 511

B. 1024

C. 510

D. 512

E. 720

Solution: Using the Binomial Theorem, $(1+1)^9 = \sum_{i=0}^{9} \binom{9}{i}(1)^i(1)^{9-i} = \sum_{i=0}^{9} \binom{9}{i}$, so $512 - \binom{9}{0} = 511$ is the desired sum.

Alternately, there are $2^9 = 512$ subsets of a set of nine objects, only one of which has no elements. Since $\binom{9}{k}$ is the set of subsets with exactly k elements, the sum given is the total number of subsets with at least one element. Hence, the sum is $512 - 1 = 511$.

29. A car has traveled for two hours at an average speed of 40 miles per hour. How fast must the car travel for the next hour so that the average speed will be 60 miles per hour for the entire three-hour trip?

A. 80 miles per hour
B. 120 miles per hour
C. 90 miles per hour
D. 100 miles per hour
E. There is no speed at which the car could travel that would result in that average speed

Solution: The distance traveled in the first two hours is 80 miles. For the average speed over three hours to be 60 miles per hour, the distance traveled must be 180 miles so that the average speed is $\frac{180}{3} = 60$ miles per hour. So, the car must travel another 100 miles in the last hour, meaning that it must travel at 100 miles per hour.

30. Find the distance from the graph of $y = \sqrt{2x}$ to the point $(3, 0)$ (meaning the smallest distance from a point on the graph to the point $(3, 0)$).

A. 3
B. $2\sqrt{2}$
C. $1 + \sqrt{2}$
D. $\sqrt{5}$
E. $\sqrt{2}$

Solution: The distance from a point on the graph of $y = \sqrt{2x}$ to $(3, 0)$ is $\sqrt{(x-3)^2 + \left(\sqrt{2x} - 0\right)^2}$. This is minimized when its square $D = (x-3)^2 + 2x$ is minimized, subject to the constraint that $x \geq 0$. Since $D = x^2 - 4x + 9$ is smallest at the vertex whose x value is 2 (which is greater than zero and falls within the constraint), the minimum distance is $\sqrt{D(2)} = \sqrt{5}$.