State Math Contest 2024 Junior Level

Instructions:

- Calculators, cell phones and other computational devices are not permitted (you can only use pens, pencils and paper to work on your answers, and then mark your answers with a number two pencil on the answer sheet).
- Correct answers are worth 5 points. Unanswered questions will be given 1 point. Incorrect answers will be worth 0 points. *This means that it will not, on average, increase your score to guess answers randomly.*
- Fill in the answers on the answer sheet using a number two pencil.
- Time limit: 120 minutes.
- When you are finished, please give the exam and any scratch paper to the test administrator.
- Good luck!

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1. Andrew sold a lamp for 5 dollars less than the listed price and received 10% of the sale revenue as a commission. Brittany sold the same type of lamp (with the same listed price) for 20 dollars less than the listed price, but she received 20% of the sale revenue as commission. Assuming there was no sales tax, if both Andrew and Brittany received the same commission then what was the listed price of the lamp in dollars?

   A. 28
   B. 32
   C. 35
   D. 40
   E. 45

2. A polynomial of degree 4 whose coefficients are real numbers has zeros $2, 3$ and $1 - i$. If the coefficient of x^4 is 1, what is the coefficient of x^3?

   A. -22
   B. -5
   C. -7
   D. 18
   E. 12

3. Let f be a function so that $f(x + y) = f(x)f(y)$ for all integers x and y. If $f(1) = 2$ then find $f(0) + f(1) + f(2) + .. + f(9)$.

   A. 1033
   B. 1023
   C. 1212
   D. 1022
   E. 1134

4. A total of 50 athletes are competing in a track and field contest, each competing in at least one of only three available events. One is the fifty meter dash, one is the hammer throw, and the last is the pole vault. There are 31 competitors in the fifty meter dash, 24 in the hammer throw and 23 in the pole vaulting competition. There are 13 in both the fifty meter dash and the hammer throw, 11 in both the fifty meter dash and the pole vault, and 9 in both the pole vault and the hammer throw. How many athletes are competing in all three events?

   A. 0
   B. 3
   C. 4
   D. 7
   E. 5

5. If x^{2024} is divided by $x - 2$ then what is the remainder?

   A. 0
   B. 1
   C. 16320
   D. 2^{2024}
   E. x

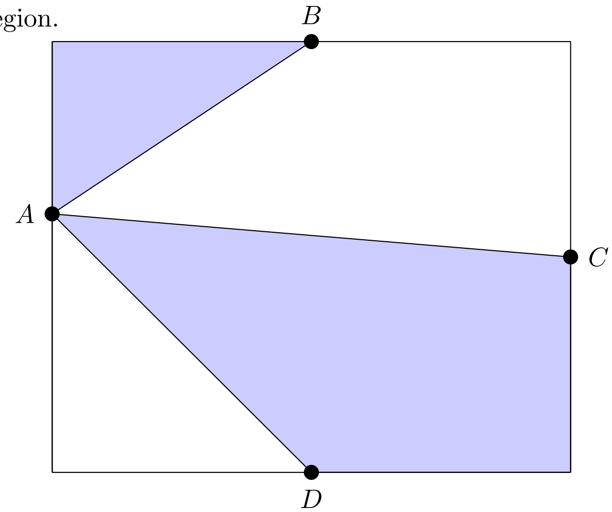
6. What is the sum of all positive integers less than 1000 that are divisible by 7 but not by 5?

   A. 70771
   B. 71071
   C. 70071
   D. 56861
   E. 92150

7. Let $f(x + y) = f(x) + f(y) + 2xy - 2$ for all real numbers x, and let $f(1) = 6$. Find $f(5)$.

   A. 25
   B. 13
   C. 48
   D. 56
   E. 42

8. The rectangle shown has an area of 36. B, C, D are the midpoints of the sides of the rectangle shown. The point A is three fifths of the distance from the bottom of the side shown. Find the area of the shaded region.



   A. 9
   B. 24
   C. 18
   D. 12
   E. 27

9. Out of a class of ten people, three people must be chosen to participate in an athletic event and a fourth person must be chosen to go along to help with the equipment and registration as a team manager. How many ways can a manager and a set of three participants be selected from the class of ten people?
   A. 240
   B. 1200
   C. 840
   D. 210
   E. 720

10. If $6x + 5y = 4$, find $64^x 32^y$.
   A. 16
   B. 32
   C. 4
   D. 64
   E. 128

11. Find the units digit of $7^{2031} \times 3^{1202} \times 6^{999}$.
   A. 0
   B. 1
   C. 2
   D. 7
   E. 8

12. A pump can fill a swimming pool in 8 hours. A second pump can fill the same pool in 10 hours. When the pool's drain is open, the (completely filled) pool drains in 20 hours. Assume both pumps are operating and the drain is open, and each pump adds water at a constant rate and water is also removed through the drain at a constant rate. How long will it take to fill the pool?

A. $5\,\frac{4}{7}$ hours

B. $5\,\frac{2}{7}$ hours

C. $5\,\frac{3}{7}$ hours

D. $5\,\frac{1}{7}$ hours

E. $5\,\frac{5}{7}$ hours

13. A fair coin is flipped three times. What is the probability that two or more heads are flipped given that one of the flips was a head?

A. $\frac{1}{2}$

B. $\frac{2}{3}$

C. $\frac{1}{3}$

D. $\frac{5}{7}$

E. $\frac{4}{7}$

14. Let $a = \sqrt{6+\sqrt{6+\sqrt{6+...}}}$, and $b = \sqrt{12-\sqrt{12-\sqrt{12-...}}}$. Find ab.

A. 9

B. 6

C. $\sqrt{48}$

D. Infinity

E. 8

15. Which of the following numbers is the smallest?

A. $2\sqrt[3]{10}$

B. $\frac{3\pi}{2}$

C. $3\sqrt{2}$

D. $\log_5 4000$

E. $\sqrt[6]{6000}$

16. Find the statement which is logically equivalent to the statement "If either of A or B are true then it follows that it is not true that both C and D are false."

    A. C is true or A is false

    B. If C is true then A is false.

    C. Either C and D are both true or A and B are both false.

    D. Either it is true that C is true or D is true, or it is true that A is false and B is false.

    E. Either it is true that C is true or D is true, or it is true that A is false or B is false.

17. Which of the following is equivalent to $\sin(2\tan^{-1}(x))$

    A. $\dfrac{x}{x^2+1}$

    B. $\dfrac{2x}{x^2+1}$

    C. $\dfrac{2x}{\sqrt{x^2+1}}$

    D. $\dfrac{x^2}{x^2+1}$

    E. $\dfrac{x}{\sqrt{x^2+1}}$.

18. Three disks of radius one are mutually externally tangent to each other. Find the area enclosed by the circumscribed equilateral triangle containing the three disks.

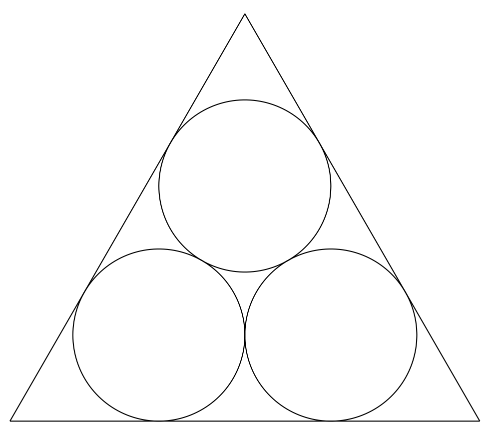
    A. 9

    B. $4+\dfrac{3\sqrt{3}}{2}$

    C. $\dfrac{9}{2}$

    D. $4\sqrt{3}+3$

    E. $6+4\sqrt{3}$



19. Find $(\log_2 5)(\log_5 7)(\log_7 16)$.

    A. $\log_{70} 560$

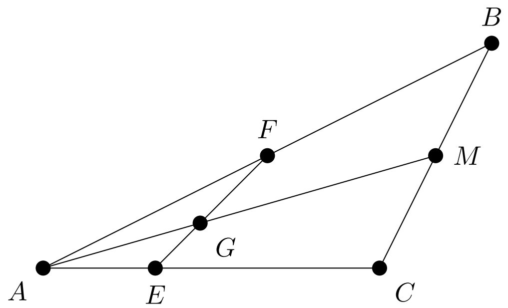
    B. 5

    C. 4

    D. 8

    E. $\log_2 560$

20. A triangle $\triangle ABC$ has side lengths $AC = 27$ and $AB = 36$. Let M be the midpoint of line segment $\overline{BC}$. Let F be a point on line segment $\overline{AB}$ and E be a point on line segment $\overline{AC}$ so that $AF = 3AE$. Let G be the point of intersection of the line segments $\overline{EF}$ and $\overline{AM}$. Find $\frac{FG}{GE}$.



A. 2
B. $\frac{3}{2}$
C. $\frac{9}{4}$
D. $\frac{4}{3}$
E. Cannot be uniquely determined

21. A certain diet requires a person to eat exactly 28 grams of fiber, 11 grams of fat, and 14 grams of protein per day. Food A contains 2 grams of fiber, 1 gram of fat, and 2 grams of protein per serving, and food B contains 3 grams of fiber, 2 grams of fat, and 2 grams of protein per serving, and food C contains 4 grams of fiber, 1 gram of fat, and 1 gram of protein per serving. If a person eats only those three foods on this diet, then how many servings of food C must the person eat each day?

A. 1
B. 2
C. 3
D. 4
E. 5

22. Anna and Bill play a game. Anna goes first and rolls a fair six-sided die. If she rolls a one or two then she wins. Otherwise, Bill goes and flips a fair coin. If he flips a head then he wins. Otherwise, Anna goes again, rolling her die and winning on a one or a two. If she does not win then Bill goes again, winning on a flip of heads with his coin and so on. What is the probability that Anna wins the game?

A. $\frac{2}{3}$
B. $\frac{1}{3}$
C. $\frac{1}{2}$
D. 0.6
E. 0.4

23. Let x, y, z be real numbers so that $\frac{2024}{x+1} + \frac{2024}{y+1} + \frac{2024}{z+1} = 2025$. Find $\frac{x-1}{x+1} + \frac{y-1}{y+1} + \frac{z-1}{z+1}$.

A. $\frac{2022}{2024}$

B. $\frac{2021}{2024}$

C. $\frac{2019}{2024}$

D. $\frac{2020}{2024}$

E. $\frac{2017}{2024}$

24. A triangle has side lengths 5, 6, and 7. Find the area enclosed by the triangle.

A. 12

B. $8\sqrt{2}$

C. $5\sqrt{3}$

D. $6\sqrt{6}$

E. 6

25. Let $n! = n(n-1)(n-2)\ldots(1)$ for any non-negative integer n. Find the number of consecutive zeros at the end of the number 250!.

A. 12

B. 50

C. 62

D. 125

E. 184

26. Let n be the positive integer written with a string of 2024 consecutive 1's for the digits, so $n = 111...11$, a string of 2024 1's. Of the following, which is the largest number that divides n?

A. 11

B. 1111

C. 11111111 (eight 1's)

D. 1111111111111111 (16 1's)

E. 11111111111111111111111111111111 (32 1's)

27. If $(8x)^{\log_2 3} - (2x)^{\log_2 9} = 0$, then what is $\log_2(x)$?

A. $\frac{1}{2}$

B. 1

C. 2

D. 4

E. Undefined

28. Let $\binom{n}{k}$ refer to $\frac{n!}{(n-k)!k!}$ if $k \leq n$ and n and k are non-negative integers (where $n! = n(n-1)(n-2)\ldots(1)$ if $n \geq 1$ and $0! = 1$). Which of the following is equal to $\binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9}$?

A. 511

B. 1024

C. 510

D. 512

E. 720

29. A car has traveled for two hours at an average speed of 40 miles per hour. How fast must the car travel for the next hour so that the average speed will be 60 miles per hour for the entire three-hour trip?

A. 80 miles per hour

B. 120 miles per hour

C. 90 miles per hour

D. 100 miles per hour

E. There is no speed at which the car could travel that would result in that average speed

30. Find the distance from the graph of $y = \sqrt{2x}$ to the point $(3, 0)$ (meaning the smallest distance from a point on the graph to the point $(3, 0)$).

A. 3

B. $2\sqrt{2}$

C. $1 + \sqrt{2}$

D. $\sqrt{5}$

E. $\sqrt{2}$