2022 State Math Competition
Senior Exam
Version A

Instructions:

- Make sure to mark the version on your answer sheet.
- Correct answers are worth 5 points. Unanswered questions will be given 2 points. Incorrect answers will be worth 0 point. This means that it is not in your best interest to guess answers unless you have eliminated some possibilities.
- Fill in the answers on the answer sheet using a pencil or pen.
- Time limit: 75 minutes.
- When you are finished, please give the exam and any scrap paper to the test administrator.
- Good luck!

1. Solve for x:

$$\log_5\left(\log_2\left(\log_3\left(2x-3\right)\right)\right)=0$$

(A) 0
(B) 6
(C) 1
(D) 3
(E) 9

2. For a positive integer n, let $f(n)$ be the number of positive factors of n. For example, since 6 has factors 1, 2, 3, and 6, $f(6)=4$. What is $f(f(f(24)))$?

(A) 4
(B) 2
(C) 8
(D) 3
(E) 1

3. Which of the following shows the numbers $2^{1/2}, 3^{1/3}, 6^{1/6}$ in increasing order?

(A) $2^{1/2}<3^{1/3}<6^{1/6}$
(B) $3^{1/3}<6^{1/6}<2^{1/2}$
(C) $6^{1/6}<3^{1/3}<2^{1/2}$
(D) $3^{1/3}<2^{1/2}<6^{1/6}$
(E) $6^{1/6}<2^{1/2}<3^{1/3}$

4. Given that $i^2=-1$, simplify $(\sqrt{3}+i)^9$.

(A) 0
(B) 512
(C) -512
(D) $512i$
(E) $-512i$

5. Define a piecewise function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{2}x & x \leq 2 \\ 2x - 3 & x \geq 2 \end{cases}.$$

The function f is bijective (one-to-one and onto). What is the inverse function to f?

(A) $f^{-1}(y) = \begin{cases} 2y & y \leq 4 \\ \frac{1}{2}y - \frac{3}{2} & y \geq 4 \end{cases}$

(B) $f^{-1}(y) = \begin{cases} 2y & y \leq 2 \\ \frac{1}{2}y + \frac{3}{2} & y \geq 2 \end{cases}$

(C) $f^{-1}(y) = \begin{cases} 2y & y \leq 1 \\ \frac{1}{2}y + 3 & y \geq 1 \end{cases}$

(D) $f^{-1}(y) = \begin{cases} 2y & y \leq 1 \\ \frac{1}{2}y + \frac{3}{2} & y \geq 1 \end{cases}$

(E) None of the above

6. At 12:00 PM two ships set out from shore. Ship A travels due east at 3 miles per hour. Ship B begins exactly one mile to the north of Ship A and travels exactly 45° due north-east at $\sqrt{2}$ miles per hour. How quickly in miles per hour is the distance between Ship A and Ship B changing at 1:00 PM?

(A) $\frac{11}{\sqrt{13}}$

(B) $\frac{3}{\sqrt{2}}$

(C) $\frac{5}{\sqrt{2}}$

(D) $\frac{3}{\sqrt{5}}$

(E) None of the above

7. Which of the following numbers is the largest?

(A) the number of ways to arrange 3 red balls and 2 blue balls in a line
(B) the number of ways to choose 3 items out of a set of 5 distinct items
(C) the number of ways to put 3 distinct items into 5 boxes
(D) the number of ways to order five distinct items
(E) the number of ways to put 5 distinct items into 3 boxes

8. How many integers n exist such that $n^2 + n + 1$ is a perfect square?

(A) 1
(B) infinitely many
(C) 3
(D) 2
(E) 0

9. A sphere is inscribed in a cube of edge length 1. Another (smaller) sphere is inscribed in the space between the larger sphere and one of the corners of the cube, so that it touches three faces of the cube, as well as the larger sphere. What is the radius of the smaller sphere?

(A) $1 - \frac{\sqrt{3}}{2}$
(B) $\frac{1}{\sqrt{3}+1}$
(C) $\frac{1}{3}$
(D) $\frac{\sqrt{3}}{4}$
(E) $\sqrt{3} - 1$

10. For real numbers x and y, define the operation $x \star y = x^2 y^2$. Which of the following is false?

(A) $x \star (y \star z) = (x \star y) \star z$, for all real x, y, z.
(B) $(x \star y)^n = x^n \star y^n$, for all real x, y and integers n.
(C) $x \star 0 = 0$, for all real x.
(D) $x \star y = y \star x$ for all real x, y.
(E) $x \star \frac{1}{x} = 1$, for all real $x \neq 0$.

11. How many positive numbers x satisfy the equation $\cos(97x) = x$?

(A) 1
(B) 15
(C) 49
(D) 31
(E) 96

12. An equilateral triangle is inscribed in a circle of radius 1. What is the area of the triangle?

(A) $\sqrt{2}$
(B) $\frac{\sqrt{3}}{4}$
(C) $\frac{1}{2}$
(D) $\frac{\sqrt{2}}{2}$
(E) $\frac{3\sqrt{3}}{4}$

13. Three different numbers are randomly selected out of the set of numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. What is the probability that their sum is greater than the sum of the remaining 6 numbers?

(A) $\frac{1}{42}$
(B) $\frac{1}{84}$
(C) $\frac{1}{14}$
(D) $\frac{3}{84}$
(E) None of the above

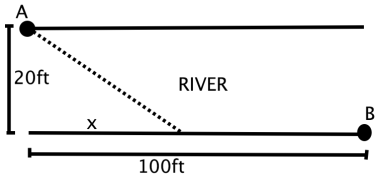
14. A number n may be written in base -2 as $a_k \cdots a_2 a_1 a_0$ where each digit a_i is 0 or 1. This means is that $n = a_k(-2)^k + \ldots + a_2(-2)^2 + a_1(-2)^1 + a_0$. For instance, 7 is 11011 when written in base -2. What is 200 written in base -2?

(A) 111101110
(B) 111011000
(C) 101101110
(D) 100110101
(E) None of the above

15. What is the remainder when 7^{111} is divided by 15?

(A) 7
(B) 13
(C) 1
(D) 4
(E) 9

16. You are on one bank of a slow-moving 20-foot wide river (labeled point A in the picture below). Suppose you want to get to point B on the other side of the river, 100 feet downstream. You want to do this as quickly as possible. You decide to first swim straight to a point x feet downstream on the opposite shore, then run the rest of the way to point B. If you run three times faster than you swim, what distance x downstream should you swim in order to get to point B as quickly as possible? Assume the current in the river is negligible.



(A) $\sqrt{\frac{80}{3}}$ feet
(B) $\sqrt{\frac{400}{3}}$ feet
(C) $\sqrt{80}$ feet
(D) $\sqrt{50}$ feet
(E) $\sqrt{\frac{400}{7}}$ feet

17. A frog is located at $(0, 0)$ in the plane. It makes a jump of length 1, meaning it jumps to a point distance 1 away from $(0, 0)$. Then it makes a second jump of length 2. What is the area of the region in the plane where the frog could land after the second jump?

(A) 8π
(B) 9π
(C) 3π
(D) 4π
(E) π

18. A set of consecutive positive integers starting with 1 is written on a board. Then one number is erased. The average (arithmetic mean) of the remaining numbers is $\frac{135}{11} = 12 + \frac{3}{11}$. What number was erased?

(A) cannot be determined
(B) 8
(C) 4
(D) 6
(E) 2

19. Watson chooses 5 distinct natural numbers and tells Sherlock the product of these numbers. However, it is not enough information for Sherlock to determine whether the sum of the chosen numbers is odd or even. Which of the following could have been the product?

(A) 1890
(B) 945
(C) 180
(D) 252
(E) 420

20. A number written in base 8 may start with some number of 0's. We call these *trailing zeroes*. For instance, the number 128 (written in decimal notation) is 200 when written in base 8; so it has 2 trailing zeroes. How many trailing zeroes does the number 100! have when written in base 8?

(A) 28
(B) 20
(C) 44
(D) 26
(E) 32