Name: ____________________

School: ____________________

Grade: ____________________

2020 State Math Contest (Senior Exam)

Weber State University

March 4, 2020

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are not allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked (a), (b), (c), (d), and (e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.
- You may not leave the room until at least 10:30 a.m.

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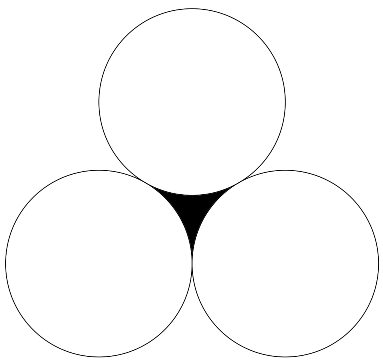
1. List all the possible values of $\frac{x}{|x|} + \frac{y+1}{|y+1|} + \frac{xz}{|xz|} + \frac{x+y+z}{|x+y+z|}$, where $x, y \in \mathbb{R}$.

   (a) $\{-4, 4\}$
   (b) $\{-4, 0, 4\}$
   (c) $\{-4, -2, 0, 2, 4\}$
   (d) All real numbers
   (e) None of the above

   Solution: (c) $\{-4, -2, 0, 2, 4\}$. $\frac{a}{|a|} = \pm 1$.

2. The radius of each of the three circles shown is r. What is the shaded area (between the circles)?

   (a) $\frac{\sqrt{3}}{2}\pi r^2$
   (b) $(\sqrt{2} - \frac{\pi}{4})r^2$
   (c) $\frac{\pi r^2}{2}$
   (d) $(\sqrt{5} - \frac{\pi}{3})r^2$
   (e) $(\sqrt{3} - \frac{\pi}{2})r^2$

   Solution: (e) $(\sqrt{3} - \frac{\pi}{2})r^2$. If we connect the centers of each of the circles, we obtain an equilateral triangle, each side of which has a length of $2r$. This triangle has a height of $r\sqrt{3}$, and thus the area of this triangle is $\frac{(r\sqrt{3})(2r)}{2} = \sqrt{3}r^2$. To find the shaded area, we take the area of the triangle, and we subtract three copies of the area of the 3 sectors of the circles with central angles $\frac{\pi}{3} \to A = \frac{1}{2}r^2(\frac{\pi}{3}) = \frac{\pi r^2}{6}$. Thus the shaded area is $\sqrt{3}r^2 - 3(\frac{\pi r^2}{6}) = \sqrt{3}r^2 - (\frac{\pi r^2}{2}) = (\sqrt{3} - \frac{\pi}{2})r^2$.

3. There are five people in a room. What is the probability at least two individuals are born in the same month?

   (a) 100%
   (b) 61.8%
   (c) 38.19%
   (d) 38%
   (e) 0%

   Solution: (b) 61.8%. The probability at least two people share the same birth month is the complementary event to no individuals share a birth month. The probability no one shares the same birth month is: $\frac{12}{12} \times \frac{11}{12} \times \frac{10}{12} \times \frac{9}{12} \times \frac{10}{12} = 0.3819$. Hence, the probability no one shares a birth month is $1 - 0.3819 = 0.6181 \approx 61.8\%$.

4. If $(1,4) \in A \cap B$ and $A = \{(x,y)|a^2x - b^2y - a = 0\}$ with $B = \{(x,y)|ax + b\sqrt{y} = 1\}$, then $A \cap B =$

   (a) $A \cap B = \{(x,y)|x = 1, y = 4\}$
   (b) $A \cap B = \{(x,y)|x = 1, y = 1\}$
   (c) $A \cap B = \{(x,y)|x = 1\}$
   (d) $A \cap B = \{(x,y)|y = 4\}$
   (e) None of the above

   Solution: (c) $A \cap B = \{(x,y)|x = 1\}$. $a^2 - 4b^2 - a = 0, a + 2b = 1 \rightarrow b = 0, a = 1$ $\rightarrow x = 1$.

5. The sides of a triangle are p, $p+1$, and $2p-1$. Its area is $2p\sqrt{10}$. Find p.

   (a) 10
   (b) 11
   (c) 21
   (d) 31
   (e) None of the above

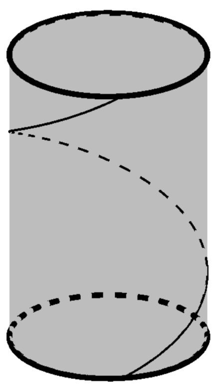
   Solution: (c) 21. Using Heron's formula, the area of the triangle is: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$. Thus, $s = \frac{(p)+(p+1)+(2p-1)}{2} = \frac{4p}{2} = 2p$. Then,
   $2p\sqrt{10} = \sqrt{[2p][2p-p][2p-(p+1)][2p-(2p-1)]} = \sqrt{(2p)(p)(p-1)(1)} = \sqrt{(2p^2)(p-1)}$.
   We now have $2p\sqrt{10} = \sqrt{(2p^2)(p-1)}$. Squaring both sides, we get
   $(4p^2)(10) = (2p^2)(p-1)$. Dividing both sides by $2p^2$, we have $20 = p - 1$, which gives
   $p = 21$.

6. If $f(x) = \frac{e^x - 1}{e^x + 1}$, then

   (a) f is odd and the range of f is $(-1, 1)$
   (b) f is odd and the range of f is $(-1, \infty)$
   (c) f is even and the range of f is $(-1, \infty)$
   (d) f is even and the range of f is $(-1, 1)$
   (e) None of the above

   Solution: (a) f is odd and the range of f is $(-1, 1)$.
   $f(-x) = -f(x) = \frac{1-e^x}{e^x+1}$
   $f(x) = \frac{e^x+1-1-1}{e^x+1} = 1 - \frac{2}{e^x+1}$
   $e^x + 1 > 1$
   $0 < \frac{2}{e^x+1} < 2$

7. A can of soup has a radius of 1 and a height of 2π. As a bug makes its way from the bottom of the can to the top, it follows the path of a helix, making one complete revolution and ending in a spot directly above where it began. (See diagram below.) How far does the bug travel?



   (a) 4π
   (b) $\frac{\sqrt{3}}{2}\pi$
   (c) $4\pi^2$
   (d) $2\sqrt{2}\pi$
   (e) $2\sqrt{3}\pi$

   Solution: (d) $2\sqrt{2}\pi$. If we imagine slicing open the cylinder with a vertical cut, we obtain a rectangular sheet with a height of 2π and a length (also equal to 2π) given by the circumference of the circle. In other words, we obtain a square whose sides each have length 2π. The hypotenuse H represents the length of the bug's path, and H can be obtained from the Pythagorean theorem: $H^2 = (2\pi)^2 + (2\pi)^2 = 8\pi^2$. Thus $H = 2\sqrt{2}\pi$.

8. What time does a 12-hour clock read 50 hours before it reads 6 o'clock?

   (a) 4 o'clock
   (b) 5 o'clock
   (c) 2 o'clock
   (d) 11 o'clock
   (e) None of the above.

   Solution: (a) 4 o'clock. If x is the time 50 hours before 6 o'clock, then $6 - 50 \equiv x \bmod(12)$, or $x \equiv -44 \equiv -44 + 4 \cdot 12 \equiv 4 \bmod(12)$.

9. Each box of cereal contains a toy. The toy comes in 4 difference colors: red, orange, green or blue. Each of the 4 colors is equally likely to occur. Alex wants to get a toy in each of the 4 colors. If Alex buys 5 boxes of cereal, what is the probability that they get at least one toy of each of the 4 colors?

   (a) $\frac{5}{1024}$
   (b) $\frac{1}{120}$
   (c) $\frac{1}{64}$
   (d) $\frac{15}{64}$
   (e) $\frac{1}{5}$

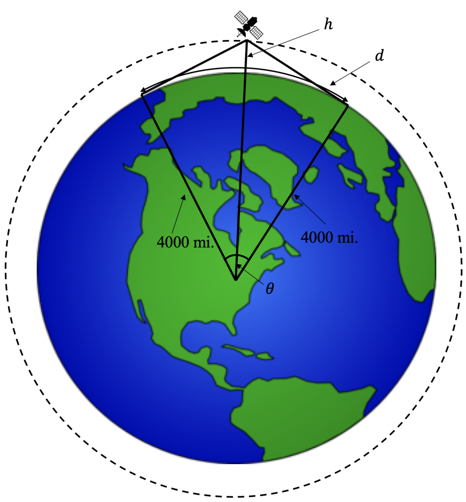
   Solution: (d) $\frac{15}{64}$. Imagine Alex opening the 5 boxes, one at a time, and recording the color of the toy from each box in order. Since there are 4 colors, there are $4 \times 4 \times 4 \times 4 \times 4 = 1024$ possible color options. To get at least one toy of each of the 4 colors, there will be exactly two boxes of the same color. There are 5 choose 2, or 10, ways to determine which two boxes contain toys of the same color, and 4 different options for the color of the matching pair. Imagine opening those two boxes first. Afterwards, open another box. There are 3 possible colors that toy could be so that all 4 colors can be obtained. The final two boxes have 2 and 1 color options, respectively, to obtain all 4 colors. So, there are $10 \times 4 \times 3 \times 2 \times 1 = 240$ possible color options with at least one toy of each color. Thus, the probability of getting at least one toy of each color is $\frac{240}{1024} = \frac{15}{64}$.

10. Suppose that $f(x) = x(x+1)(x+2)(x+3)\cdots(x+50)$. What is the coefficient of x^{50}?

    (a) 1275
    (b) 1350
    (c) 1375
    (d) 1400
    (e) 1425

    Solution: (a) 1275. Expanding $x(x+1)(x+2)(x+3)\cdots(x+50)$ yields $x^{51}+cx^{50}+\cdots$, where the value of c is $0+1+2+\cdots+50 = \frac{50(51)}{2} = 1275$.

11. A surveillance satellite circles Earth at a height of h miles above the surface. Suppose that d is the distance, in miles, on the surface of Earth that can be observed from the satellite. See the following diagram below, which is not to scale.



    If the satellite is tasked to observe a distance (d) of 4,186.67 miles, approximately how high is the satellite above Earth's surface (h)?

    (a) 50 miles
    (b) 100 miles
    (c) 300 miles
    (d) 500 miles
    (e) 700 miles

    Solution: (e) 700 miles. The distance (d) is a fraction of Earth's circumference, which has a radius of 4,000 miles. Thus, $C = 2\pi r \to (2)(3.14)(4{,}000) \to 25{,}120$ miles. Next, we find the fraction of the circumference, which is d, to be $\to 4186.67/25{,}120 = \frac{1}{6}$. We can determine that θ is $\frac{1}{6}$ of $360°$, which is $60°$. To find the height of the satellite above Earth, we can use $\cos\frac{\theta}{2} = \cos 30° = \frac{\sqrt{3}}{2} = \frac{4{,}000}{4{,}000+h}$. Solving the proportion, we get $\sqrt{3}(4{,}000+h) = 2(4{,}000) \to \sqrt{3}\cdot 4{,}000 + \sqrt{3}\cdot h = 8{,}000$, which is approximately $1.7\cdot 4{,}000 + 1.7\cdot h = 8{,}000 \to 6{,}800 + 1.7\cdot h = 8{,}000 \to 1.7\cdot h = 1{,}200$, which is approximately 700 miles. (Actual distance is 612.6 miles.)

12. A company has a storage tank for vehicle fuel. The tank is a right circular cylinder with its axis parallel to the ground. The radius is 2 feet and the length is 8 feet. If the tank is full on day 1 and the depth on day 20 is 1 foot, then determine the average daily consumption to the nearest tenth of a ft^3/day.

(a) 3.8 ft^3/day

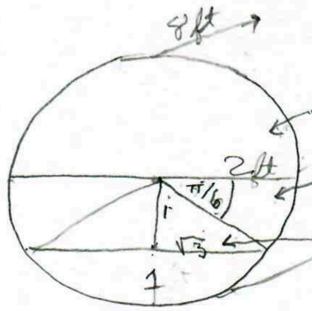
(b) 4.0 ft^3/day

(c) 4.5 ft^3/day

(d) 4.8 ft^3/day

(e) 5.1 ft^3/day

Solution: (b) 4.0 ft^3/day.



$A_1 = \frac{1}{2}\pi 4 = 2\pi$

$A_2 = \frac{1}{2}\cdot 4 \cdot \frac{\pi}{6} = \frac{\pi}{3}$

$A_3 = \frac{1}{2}\sqrt{3}\cdot 1 = \frac{\sqrt{3}}{2}$

Volume used $V = 8(A_1 + 2A_2 + 2A_3)$

$= 8(2\pi + 2\frac{\pi}{3} + \sqrt{3})$

$= 8(8\frac{\pi}{3} + \sqrt{3})$ ft^3

≈ 81 ft^3

Daily Consumption $\frac{81}{20}$ ft^3/day $= 4.0$ ft^3/day

13. A 1-meter measuring stick is cut at a randomly selected location, creating two sticks. What is the probability that the larger of the two sticks is over three times as long as the shorter of the two sticks, given that the longer of the two sticks is over twice as long as the shorter of the two sticks?

    (a) $\frac{1}{4}$
    (b) $\frac{1}{3}$
    (c) $\frac{1}{2}$
    (d) $\frac{2}{3}$
    (e) $\frac{3}{4}$

    Solution: (e) $\frac{3}{4}$. The question is asking for a conditional probability. The probability of A given B is equal to the probability of A and B divided by the probability of B. First, determine the probability that the "larger of the two sticks is over three times as long as the shorter of the two sticks" and "the longer of the two sticks is over twice as long as the shorter of the two sticks". Both conditions are satisfied when larger of the two sticks is over three times as long as the shorter of the two sticks. To compute this probability, imagine the meter stick labeled with markings from 0 to 1. If the cut is made at any location between 0 and $\frac{1}{4}$, the smaller stick will be less than $\frac{1}{4}$ meter and the larger stick would be larger than $\frac{3}{4}$, satisfying the criteria. The same will be true if the cut is made at any location between $\frac{3}{4}$ and 1. Thus, in $\frac{1}{2}$ of the possible cut locations, the criteria is met. Next, determine the probability that "the longer of the two sticks is over twice as long as the shorter of the two sticks". This will be satisfied if the cut is made between 0 and $\frac{1}{3}$ or if the cut is made between $\frac{2}{3}$ and 1. Thus, in $\frac{2}{3}$ of the possible cut locations, the criteria is met. The conditional probability is $\frac{1}{2}$ divided by $\frac{2}{3}$, which equals $\frac{3}{4}$.

14. Suppose that $(x, y) = (5, 8)$ and $(x, y) = (3, 4)$ are solutions to the linear system of equations $\left\{\begin{array}{lcl} ax + by & = & e \\ cx + dy & = & f \end{array}\right\}$, where a, b, c, d, e, and f are real numbers. Which of the following is also a solution to this same system of equations?

    (a) $(6, 6)$
    (b) $(2, 8)$
    (c) $(4, 7)$
    (d) $(7, 12)$
    (e) $(6, 9)$

    Solution: (d) $(7, 12)$. Consider the point $(2, 4) = (5 - 3, 8 - 4)$. We can see that $(x, y) = (2, 4)$ satisfies the system $\left\{\begin{array}{lcl} ax + by & = & 0 \\ cx + dy & = & 0 \end{array}\right\}$. Then any point of the form $(5 + 2k, 8 + 4k)$ will be a solution to the system. In particular, taking $k = 1$, we obtain the solution $(7, 12)$.

15. Simplify the following: log(log(log(googolplex)))

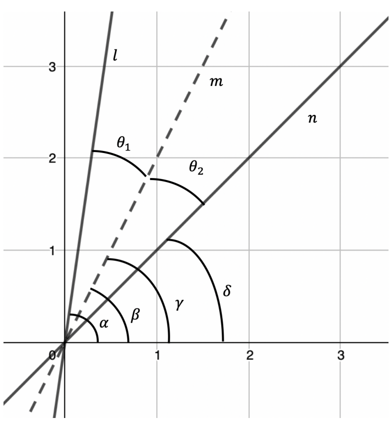
    (a) 2
    (b) 3
    (c) 10
    (d) 1 million
    (e) A googol

    Solution: (a) 2. A *googolplex* is 10^{googol}. A *googol* is 10^{100}. Thus, simplifying log(log(log(googolplex))), we get $\log(\log(\log(10^{10^{100}}))) = \log(\log(10^{100})) = \log(100) = 2$.

16. An acute angle is formed by two lines of slope 1 and 7. What is the slope of the line which bisects this angle?

    (a) 2
    (b) 3
    (c) 1
    (d) 4
    (e) None of the above

    Solution: (a) 2. Note the following diagram:



    Line l has a slope of 7 and can be set to $y = 7x$. Line n has a slope of 1 and can be set to $y = x$. We need to find the slope of line m, which bisects the angle between line l and line n, i.e., find the slope of a line such that $\theta_1 = \theta_2$. Let $\theta_1 = \alpha - \beta$ and let $\theta_2 = \gamma - \delta$. Using the formula for the tangent of the difference of two angles, we find $\tan(\theta_1) = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{7-x}{1+7x}$. Similarly, we find $\tan(\theta_2) = \tan(\gamma - \delta) = \frac{\tan\gamma - \tan\delta}{1 + \tan\gamma\tan\delta} = \frac{x-1}{1+x}$. Setting $\theta_1 = \theta_2$, we obtain $\frac{x-1}{1+x} = \frac{7-x}{1+7x}$, where x is the desired slope. Solving this equation gives $x = 2$ and $x = -\frac{1}{2}$.

17. Washington School sent their 4 best chess players to challenge the 4 best chess players from Lincoln School. Each of the 4 Washington players will be randomly paired with a different Lincoln player for the first round of games. How many different possible pairings are possible for the first round of games?

(a) 4
(b) 16
(c) 24
(d) 576
(e) 40,320

Solution: (c) 24. Imagine that there are 4 tables with chessboards set up, and that each of the Lincoln players is already seated at a table when the Washington players arrive. The order of the Lincoln players is unimportant. The first Washington player is randomly assigned to one of the 4 seated Lincoln players. The next Washington player is randomly assigned to one of the 3 remaining Lincoln players. The next Washington player is assigned to one of the 2 remaining Lincoln players. And the last Washington player has only one choice of Lincoln opponent. Therefore, the total number of pairings is $4 \times 3 \times 2 \times 1 = 24$.

18. Find the exact value of $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

(a) 1
(b) 2
(c) -1
(d) 3
(e) 6

Solution: (a) 1. Let $x = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$. Then,
$x^3 = 2+\sqrt{5}+3\sqrt[3]{(2+\sqrt{5})^2(2-\sqrt{5})}+3\sqrt[3]{(2+\sqrt{5})(2-\sqrt{5})^2}+2-\sqrt{5}$
$= 4+3\sqrt[3]{(2+\sqrt{5})(4-5)}+3\sqrt[3]{(4-5)(2-\sqrt{5})} = 4-3x$. The equation $x^3+3x-4=0$ has a solution $x=1$ and so it can be factored: $(x-1)(x^2+x+4)=0$. Since the quadratic equation $x^2+x+4=0$ has no real-valued solution, we must have $x=1$.

19. An octahedron is formed by connecting the centers of the faces of a cube. What is the ratio of the volume of the cube to that of the contained octahedron?

    (a) 4 : 1
    (b) 6 : 1
    (c) 3 : 2
    (d) 5 : 1
    (e) None of the above

    Solution: (b) 6 : 1. Let the side length of the cube equal 1 unit. The octahedron is the union of two pyramids of height $\frac{1}{2}$ on a base which is a square of side length $\frac{\sqrt{2}}{2}$. The volume of each pyramid is $\frac{1}{12}$.

20. The police were informed of a car that was speeding. The license plate had six entries that could be either numbers or letters. The person reporting the car said that the first three entries of the license plate were digits and the last three were letters. The person saw that the first digit was 8. The last three letters comprised of "WHY", but the person could not remember the order of the letters. How many license plates must the police check?

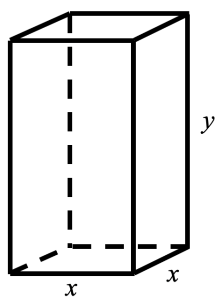
    (a) 100
    (b) 600
    (c) 6
    (d) 60
    (e) 300

    Solution: (b) 600. There are $3! = 6$ ways to rearrange the order of the letters, and there are $10 \times 10 = 100$ possible digits to fill in the 2nd and 3rd digit of the license plate. Hence, by the multiplication rule there are $6 \cdot 100 = 600$ different license plates.

21. A cardboard box has a square base, with each edge of the base having a length of x inches. The total length of all 12 edges is 16 inches. How long should x be to maximize the volume?

(a) $\frac{1}{3}$ inches
(b) $\frac{2}{3}$ inches
(c) 1 inch
(d) $\frac{4}{3}$ inches
(e) None of the above

Solution: (d) $\frac{4}{3}$ inches. Each edge of the square base has a length of x inches and the height of the box is y inches as shown below:



Since the lengths of all the edges sum to 16 inches, we know that $8x + 4y = 16$ and then $y = 4 - 2x$. The volume is $V = x^2y$, which with substitution can be written as $x^2(4 - 2x) = 4x^2 - 2x^3$. In order to maximize volume, we can take the derivative of the volume function, set it equal to zero, and solve for x. Thus, $\frac{dV}{dx} = 8x - 6x^2 \rightarrow 8x - 6x^2 = 0 \rightarrow 2x(4 - 3x) = 0 \rightarrow x = 0, \frac{4}{3}$. We use the second derivative to identify if $x = 0$ and $x = \frac{4}{3}$ are maxima or minima. $\frac{d^2V}{dx^2} = 8 - 12x$. Thus, $x = 0$ is positive and is a minumum and at $x = \frac{4}{3}$ is negative, which gives us the maximum.

22. Convert the base 7 number 2641.53 to base 10. What is the digit in the tenths position?

(a) 3
(b) 4
(c) 6
(d) 7
(e) 8

Solution: (d) 7. Convert 0.53 base 7 to base 10: $\frac{5}{7} + \frac{3}{7^2} = \frac{5 \cdot 7+3}{49} = \frac{38}{49} = .7755...$, so the tenths digit is 7.

23. A family just moved into a new house with an octagon pool. It is 10 feet deep and 20 feet across (from vertex to opposite vertex). The family wants to fill it up, but needs to know how much water it will take. What is the volume of the pool?

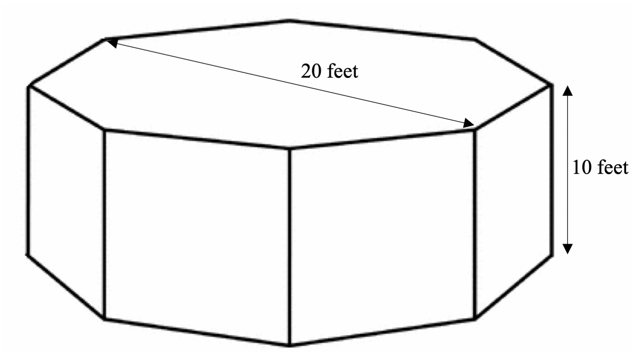
(a) $1000\sqrt{2}$ ft^3

(b) $2000\sqrt{2}$ ft^3

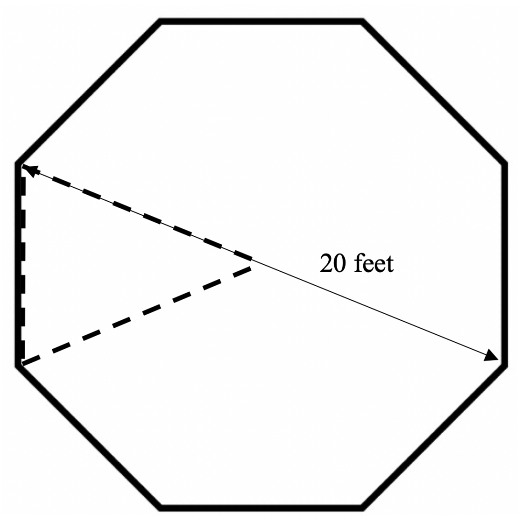
(c) $4000\sqrt{2}$ ft^3

(d) $8000\sqrt{2}$ ft^3

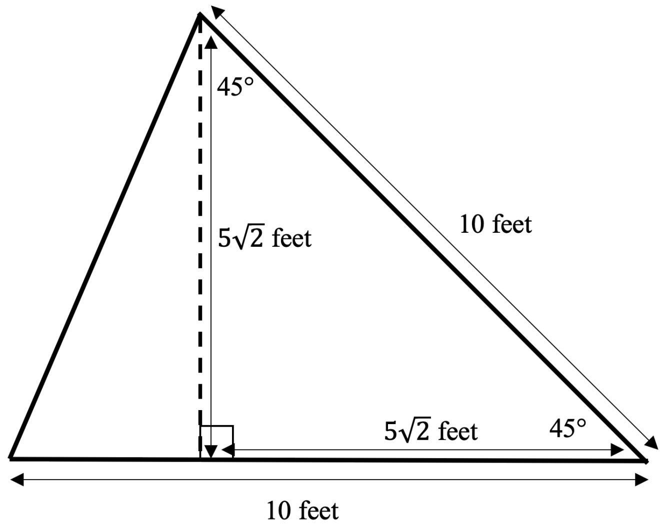
Solution: (b) $2000\sqrt{2}$ ft^3. Volume can be found by multiplying the area of the base by the height. The following diagram displays the dimensions:



The base of the octagonal prism is depicted in the following diagram:



The dashed triangle represents $\frac{1}{8}$ of the octagon. Therefore, the area of the base is the area of the triangle multiplied by 8. The dimensions of the triangle are found in the following diagram:



The central angle of an octagon is 45°, thus by finding the height of the triangle we can create a 45°-45°-90° special right triangle. Since the hypotenuse of this triangle is 10 feet, then the length of the leg is $\frac{10}{\sqrt{2}} \to \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \to \frac{10\sqrt{2}}{2} \to 5\sqrt{2}$ feet. Thus, the area of the triangle is $\frac{1}{2} \cdot base \cdot height = \frac{1}{2}(10\text{ ft})(5\sqrt{2}\text{ ft}) = 25\sqrt{2}\text{ ft}^2$. Multiplying this number by 8, we get the area of the base to be $200\sqrt{2}\text{ ft}^2$. Multiplying this number by the height of 10 ft, we get the volume to be $2000\sqrt{2}\text{ ft}^3$.

24. A student has a row of books that she is interested in reading. There are 3 books on skiing, 5 on mathematics, and 4 books on statistics. She would like to keep these books together on a single row on her book shelf. The mathematics and statistics books should be kept back to back, but we may permute the individual books within topics. How many different permutations of the books are possible?

    (a) 103,680
    (b) 17,280
    (c) 69,120
    (d) 34,560
    (e) 17,280

    Solution: (c) 69,120. There are 2 ways to keep the skiing books together, and there are two ways to keep the Math and Statistics books back to back. There are 3!=6 ways to permute the skiing books, there are 5!=120 ways to permute the math books, and there are $4! = 24$ ways to rearrange the statistics books. Hence, there are $2 \times 2 \times 6 \times 120 \times 24 = 69,120$ permutations.

25. In triangle ABC, $AC = 6$ and $BC = 5$. Point D on AB divides it into segments of length $AD = 1$ and $DB = 3$. What is the length CD?

(a) 11

(b) $\frac{15}{2}$

(c) $\frac{11}{2}$

(d) 15

(e) None of the above

Solution: (c) $\frac{11}{2}$. If x is the desired length and $\theta = \angle BDC$, then we have $x^2 + 1 + 2x\cos\theta = 36$ and $x^2 + 9 - 6x\cos\theta = 25$. Adding 3 times the first equation to the second yields $4x^2 + 12 = 133$.

26. Suppose $y'' + y = 0$. Which of the following is a possibility for $y = f(x)$?

(a) $y = \tan x$

(b) $y = \sec x$

(c) $y = e^x$

(d) $y = \sin x$

(e) $y = \frac{1}{x}$

Solution: (d) $y = \sin x$. Notice that $\frac{d^2}{dx^2}(\sin x) = -\sin x$, and $-\sin x + \sin x = 0$.

27. What is the number of real-valued solutions of the equation $x^2 + 21 = 10|x|$.

(a) 4

(b) 2

(c) 0

(d) 6

(e) None of the above

Solution: (a) 4. For $x \geq 0$, the equation $x^2 - 10x + 21 = 0$ has two solutions $x = 3, 7$. For $x \leq 0$, the equation $x^2 + 10x + 21 = 0$ has two solutions $x = -7, -3$.

28. Let n be greatest integer that will divide the three integers 15482, 15678, and 16168 and leave the same remainder. Which is the sum of the digits of n?

    (a) 5
    (b) 17
    (c) 18
    (d) 21
    (e) 25

    Solution: (b) 17. Dividing each with remainder means for each number, $x - r = qn$, where n is the number we want and q is the quotient, r the remainder. Hence n divides each of the differences of the numbers. So n is the greatest common divisor of $16168 - 15678 = 490 = 2 \cdot 5 \cdot 7^2$, $16168 - 15482 = 686 = 2 \cdot 7^3$, and $15678 - 15482 = 196 = 2^2 \cdot 7^2$. Hence the GCD is $2 \cdot 7^2 = 98$ and $9 + 8 = 17$.

29. A person's blood pressure, P, varies with the cycle of their heartbeat. The pressure (in mmHg) at time t seconds for a particular person may be modeled by the function: $P(t) = 100 + 20\cos(2\pi t)$ mmHg, $t \geq 0$. According to this model, which of the following statements is accurate?

    (a) The maximum pressure is 100 mmHg.
    (b) The pressure goes through one complete cycle in 2 seconds.
    (c) The amplitude of the pressure function is 120 mmHg.
    (d) The pressure will reach a maximum value at time $t = 1$ second.
    (e) Both statements (b) and (d) are accurate.

    Solution: (d) The pressure will reach a maximum value at time $t = 1$ second. The period is $\frac{2\pi}{2\pi} = 1$. Since cosine reaches its maximum at the end of each period, the function will reach a maximum value at time $t = 1$ second.

30. Let f be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(b) = 10$ and $f(a) = 2$. On which of the following intervals $[a, b]$ would the Mean Value Theorem guarantee a $c \in (a, b)$ such that $f'(c) = 4$?

    (a) $[0, 4]$
    (b) $[0, 3]$
    (c) $[2, 4]$
    (d) $[1, 10]$
    (e) $(0, \infty)$

    Solution: (c) $[2, 4]$. $\frac{f(b)-f(a)}{b-a} = \frac{10-2}{4-2} = 4$.

31. Two roots of the polynomial $3x^3 + \alpha x^2 - 5x - 10$ are r and $-r$ for some real number r. What is the value of α?

    (a) 1
    (b) 2
    (c) -1
    (d) 3
    (e) 6

    Solution: (e) 6. We must have $3x^3 + \alpha x^2 - 5x - 10 = (x^2 - r^2)(3x + \frac{10}{r^2})$. The second factor is chosen to make the constant and cube terms work out. Thus, $\alpha = \frac{10}{r^2}$ and $-5 = -3r^2$. Hence, $\alpha = \frac{10}{\frac{5}{3}} = 6$.

32. A ship has spotted an island and is approaching its midpoint. It's distance from the island is about 9,130 meters and the angle it subtends is 120°. What is the estimated width of the island?

    (a) 4,565 meters
    (b) 9,130 meters
    (c) 18,260 meters
    (d) $9{,}130\sqrt{3}$ meters
    (e) $18{,}260\sqrt{3}$ meters

    Solution: (e) $18{,}260\sqrt{3}$ meters. Let d be the width of the island. Set $\tan 60^\circ = \frac{\frac{d}{2}}{9130}$. Then $\frac{d}{2} = 9130 \cdot \tan 60^\circ \to d = 2(9130 \cdot \sqrt{3}) \to 18{,}260\sqrt{3}$ meters.

33. Find the length of the curve $x = 3t^2$, $y = 2t^3$, $0 \le t \le 1$.

(a) $2\sqrt{2} - 2$

(b) $4\sqrt{2} - 2$

(c) $4\sqrt{2}$

(d) $4\sqrt{2} - 1$

(e) None of the above.

Solution: (b) $4\sqrt{2} - 2$.
$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$
$\rightarrow \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$
$\rightarrow \int_0^1 6t\sqrt{1 + t^2}\, dt$
$\rightarrow 6 \cdot \int_0^1 t\sqrt{(1 + t^2)}\, dt$.
Let $u = 1 + t^2$, then $du = 2t\, dt \rightarrow \frac{du}{2} = t\, dt$
$\rightarrow 6 \cdot \int \frac{\sqrt{u}}{2}\, du$
$\rightarrow 3 \cdot \int u^{\frac{1}{2}}\, du$
$\rightarrow 3 \cdot \frac{2}{3} u^{\frac{3}{2}}$
$\rightarrow 2 \cdot (1 + t^2)^{\frac{3}{2}} \Big|_0^1$
$\rightarrow 2\Big([1 + (1)^2]^{\frac{3}{2}} - [1 + (0)^2]^{\frac{3}{2}}\Big)$
$\rightarrow 2 \cdot 2^{\frac{3}{2}} - 2 \cdot 1^{\frac{3}{2}}$
$\rightarrow 4\sqrt{2} - 2$

34. Find exact value of the product $\log_c a \cdot \log_a b \cdot \log_b c$, if $\log_c b = 3$.

(a) 3

(b) 27

(c) 9

(d) 1

(e) None the above.

Solution: (d) 1. Due to the properties of logarithms:
$\log_c a \cdot \log_a b \cdot \log_b c = \log_c a \cdot \log_a b^{\log_b c} = \log_c a \cdot \log_a c = \log_c a^{\log_a c} = \log_c c = 1.$

35. What is the exact value of $2(\sec\theta - \tan\theta)^2$, if $\frac{1-\sin\theta}{1+\sin\theta} = 3$?

(a) 3
(b) 9
(c) 1
(d) 5
(e) None of the above

Solution: (e) None of the above. Since $\frac{1-\sin\theta}{1+\sin\theta} = \frac{1-\sin\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta} = \frac{(1-\sin\theta)^2}{1-\sin^2\theta} = \frac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}$ $= \frac{1}{\cos^2\theta} - 2\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta = (\sec\theta - \tan\theta)^2$. Then $2(\sec\theta - \tan\theta)^2 = 2\cdot 3 = 6$.

36. Find the sum of the following infinite series: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$.

(a) 0.5
(b) 1
(c) 2
(d) 0.968
(e) 0.937

Solution: (b) 1. This is a geometric series with $r = 0.5$ and $a = 0.5$. The sum of a geometric series of this form is $\frac{a}{1-r} = 1$.

37. What is the greatest whole number that is a factor of the sum of any four consecutive positive even integers?

(a) 1
(b) 2
(c) 4
(d) 8
(e) 16

Solution: (c) 4. The sum of 4 consecutive even integers is $2n + (2n+2) + (2n+4) + (2n+6) = 8n + 12 = 4(2n+3)$. Hence 4 is the largest factor that will work for all positive n, since $2n+3$ will range over all odd numbers greater than or equal to 5, which don't all have common factors.

38. Suppose that $2x + \sqrt{2x + \sqrt{2x + \sqrt{2x + \cdots}}} = 2$, then x is

(a) $2 + \sqrt{2}$
(b) $2 - \sqrt{2}$
(c) $2 - \sqrt{3}$
(d) $2 + \sqrt{3}$
(e) None of the above.

Solution: (e) None of the above.
$\sqrt{2x + \sqrt{2x + \sqrt{2x + \cdots}}} = 2 - 2x$
$\to 2x + \sqrt{2x + \sqrt{2x + \sqrt{2x + \cdots}}} = (2 - 2x)^2$
$\to 2 = 4 - 8x + 4x^2$
$\to 0 = 2 - 8x + 4x^2$
$\to x = \frac{2 \pm \sqrt{2}}{2}$

39. The two shortest sides of a right triangle have lengths 3 and $\sqrt{7}$. Let α be the smallest angle of the triangle. What is $\cos \alpha$?

(a) $\frac{3}{4}$
(b) $\frac{\sqrt{7}}{3}$
(c) $\frac{\sqrt{7}}{4}$
(d) $\frac{2}{3}$

Solution: (a) $\frac{3}{4}$. The hypotenuse length is $\sqrt{9+7} = 4$. Angle α is opposite the shortest side, so the adjacent side has length 3, so $\cos \alpha = \frac{3}{4}$.

40. How many pairs of positive integers (x, y) satisfy the equation $2x + 11y = 340$?

(a) 15
(b) 10
(c) 20
(d) 5

Solution: (a) 15. Since both $2x$ and 170 are even numbers and factor 11 is odd, y must be an even number, $y = 2z$. Then the equation reduces to $x + 11z = 170$. Possible positive integer z values are $1, 2, \cdots, 15$ and for each such z value there is a unique corresponding positive integer value x.