Name: _____

School: _____

Grade: _____

2019 State Math Contest (Senior Exam)

Weber State University

March 5, 2019

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are not allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked (a), (b), (c), (d), and (e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.
- You may not leave the room until at least 10:15 a.m.

---

1. Using each of the nine digits (1, 2, 3, 4, 5, 6, 7, 8, 9) only once, find the smallest sum that primes, created from the nine digits, can form. For example, 61 + 283 + 47 + 59 = 450.

   (a) 113
   (b) 207
   (c) 321
   (d) 431
   (e) None of the above

2. Let $f(x) = ax^2 + bx + c$ be a quadratic function with two real roots, x_1 and x_2. If $f(x_1) = p$, then $f'(x_2)$ is:

   (a) p
   (b) $-p$
   (c) $2ap + b$
   (d) $-2ap + b$
   (e) None of the above

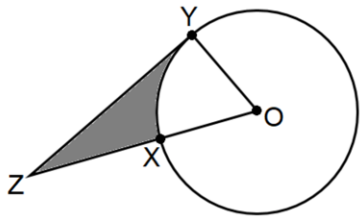
3. What is the coefficient of the term which contains y^4 in the expansion of $(2x^2 - \sqrt{y})^{14}$?

   (a) 192,192
   (b) 256
   (c) $-439,296$
   (d) $-262,144$
   (e) 1,025,024

4. A jar with one marble sits on a desk. The first student passes the jar and adds a marble. The second student passes the jar and adds two marbles. The third student adds three marbles. Each student that passes adds one more marble to the jar than the previous student added. How many marbles will be in the jar after 100 students have added marbles to the jar?

   (a) 101
   (b) 501
   (c) 1,001
   (d) 5,051
   (e) 10,101

5. In the figure below, O is the center of the circle, $\overline{YZ}$ is tangent to the circle at Y, and X lies on $\overline{OZ}$. If $\overline{OX} = \overline{XZ} = 6$, what is the area of the shaded region?



(a) $18\sqrt{3} - 3\pi$
(b) $18\sqrt{3} - 6\pi$
(c) $36\sqrt{3} - 3\pi$
(d) $36\sqrt{3} - 6\pi$
(e) $18\sqrt{3} - 2\pi$

6. How many different numbers can you get by adding three different numbers from the set {3, 6, 9, 12, ..., 21, 24}?

(a) 15
(b) 16
(c) 18
(d) 20
(e) 22

7. A hot air balloon rising straight up from a level field is tracked by a rangefinder 150 meters from the liftoff point. At the moment the rangefinder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.15 radians/minute. How fast is the balloon rising at that moment?

(a) $22.5\sqrt{2}$ meters/min
(b) 45 meters/min
(c) $\frac{45}{\sqrt{2}}$ meters/min
(d) 150 meters/min
(e) 0.3 meters/min

8. Given that $x^2 + y^2 = 9$, what is the largest possible value of $x^2 + 3y^2 + 4x$?

(a) 22
(b) 24
(c) 26
(d) 29
(e) 36

9. A 50-inch piece of wire is cut into two pieces which are then bent into a square and a circle. How long should the two pieces be in order to minimize the sum of the areas?

   (a) 25 inches, 25 inches
   (b) 20 inches, 30 inches
   (c) 38 inches, 12 inches
   (d) 4 inches, 46 inches
   (e) 22 inches, 28 inches

10. A **cycloid** is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping.

    One arch of a cycloid generated by a circle of radius r can be parameterized by

    $x = r(t - \sin t)$
    $y = r(1 - \cos t)$,

    where t is the angle through which the rolling wheel has rotated.

    Find the area under one cycloid cycle for a wheel with radius 18 inches.

    (a) 4π feet squared
    (b) $\frac{21\pi}{4}$ feet squared
    (c) 6π feet squared
    (d) $\frac{27\pi}{4}$ feet squared
    (e) 8π feet squared

11. Let $\log_b x = 2$, $\log_b y = 4$, and $\log_b z = 8$. For some positive values of a, b, x, y, and z, what is the exact value of $\left(\log_{a^4} \frac{x^2y^3}{\sqrt{z}}\right) \log_b a$?

    (a) 1
    (b) 2568
    (c) -7
    (d) 3
    (e) None of the above

12. **Trailing zeros** are a sequence of zeros after which no other nonzero digits follow. Determine how many trailing zeros are in (130!)

    (a) 13
    (b) 26
    (c) 31
    (d) 32
    (e) 35

13. What are the dimensions of a right circular cylinder can that will contain 1 cubic foot of liquid and that will have the smallest amount of surface area?

    (a) The ideal radius is between 0.3 feet and 0.4 feet
    (b) The ideal radius is between 0.4 feet and 0.5 feet
    (c) The ideal radius is between 0.5 feet and 0.6 feet
    (d) The ideal radius is between 0.6 feet and 0.7 feet
    (e) None of above

14. A certain Archimedean solid built from squares and triangles, known as a rhombicuboctahedron, has 24 vertices and 48 edges. How many faces does it have?

    (a) 72
    (b) 1152
    (c) 2
    (d) 26
    (e) 36

15. What is the exact value of $\cos\left(\sec^{-1}(\frac{3}{2}) + \tan^{-1}(-\frac{1}{4})\right)$?

    (a) $\frac{3}{2} - \frac{1}{4}$
    (b) $\frac{8\sqrt{17}+\sqrt{5}\sqrt{17}}{51}$
    (c) $\frac{\sqrt{5}\sqrt{17}-8\sqrt{17}}{51}$
    (d) $\frac{8\sqrt{17}-\sqrt{5}\sqrt{17}}{51}$
    (e) None of the above

16. What is the remainder when 2^{1000} is divided by 13?

    (a) 2
    (b) 3
    (c) 4
    (d) 10
    (e) 12

17. An integer when divided by 2 leaves a remainder of 1, when divided by 3 leaves a remainder of 2, when divided by 4 leaves a remainder of 3, and when divided by 5 leaves a remainder of 4. Find the smallest such integer.

    (a) 29
    (b) 39
    (c) 59
    (d) 119
    (e) None of the above

18. The following trigonometric function models the position of a rung on a waterwheel:

$$y = -20\sin(\tfrac{\pi}{6}t) + 16$$

where t is in seconds and y is the number of feet above water level. What is the maximum height of any given rung?

(a) 12 feet
(b) 16 feet
(c) 20 feet
(d) 36 feet
(e) 40 feet

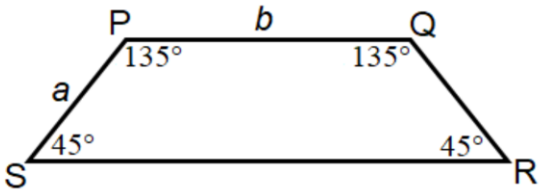
19. If one line is drawn in the (Euclidean) plane, it divides the plane into two regions. If two lines are drawn in the plane, they divide the plane into either three regions or four regions. What is the largest number of regions into which the plane can be divided if 10 lines are drawn?

(a) 10
(b) 11
(c) 20
(d) 36
(e) 56

20. Suppose you are playing basketball outside and the ball bounces off of a nail in the road. The ball begins leaking air and you notice that the diameter of the ball is shrinking at a constant rate of 4 centimeters per minute. If the diameter of the ball was 24 centimeters when you began playing, how fast is the volume of the ball shrinking 2 minutes after hitting the nail?

(a) -800π cm^3/min
(b) -24π cm^3/min
(c) -256π cm^3/min
(d) -512π cm^3/min
(e) -1024π cm^3/min

21. Trapezoid $PQRS$ is shown below. How much greater is the area of this trapezoid than the area of a parallelogram with side lengths a and b and base angles of 45°and 135°?



(a) $\frac{1}{2}a^2$
(b) $\sqrt{2}a^2$
(c) $\frac{1}{2}ab$
(d) $\sqrt{2}ab$
(e) $2a$

22. If three fair dice are rolled, what is the probability that the sum of the three dice is 6?

(a) $\frac{5}{108}$
(b) $\frac{1}{18}$
(c) $\frac{1}{24}$
(d) $\frac{10}{36}$
(e) $\frac{1}{27}$

23. Let a and b be positive numbers not equal to 1 such that $ab = a^b$ and $a^{2b} = \frac{a}{b}$. What is the value of $8a + 3b$?

(a) 27
(b) 29
(c) 30
(d) 32
(e) None of the above

24. Simplify the following expression:

$$(\sin\theta + \cos\theta)^2$$

(a) $1 + \sin 2\theta$
(b) $\cos 2\theta$
(c) $2\sin\theta\cos\theta - 1$
(d) $1 + \cos 2\theta$
(e) None of the above

25. Suppose $3x - y = 12$. What is the value of $\frac{27^x}{3^y}$?

(a) 3^{12}
(b) 9^4
(c) 27^2
(d) 3^8
(e) The value cannot be determined from the information given

26. Suppose that the number m is the product of all the positive prime numbers less than 40. How many positive divisors does m have?

(a) 412
(b) 24
(c) 4,096
(d) 37
(e) 1,600

27. What is the probability that a solution for $x^2 + x < 42$ is also a solution for $x^2 - 14x > 15$?

(a) $\frac{6}{13}$
(b) $\frac{7}{13}$
(c) $\frac{2}{13}$
(d) $\frac{1}{13}$
(e) None of the above

28. A **palindrome** is a positive integer which reads the same forwards as backwards. For example, 131 or 54845. What integer greater than 1 is a factor of every four-digit palindrome?

(a) 6
(b) 11
(c) 111
(d) 121
(e) None of the above

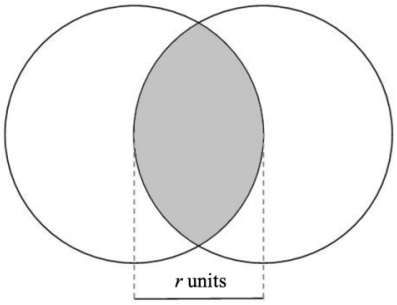
29. The number of cubic units in the volume of certain cube is half as many as the number of square units in its surface area. Find the volume of the cube (in cubic units).

(a) 9
(b) 27
(c) 36
(d) 108
(e) 206

30. A snack pack of Starbursts contains two candies. The candies might be pink, yellow, orange, or red. Each color is equally likely to appear. What is the probability of opening a snack pack with at least one yellow candy?

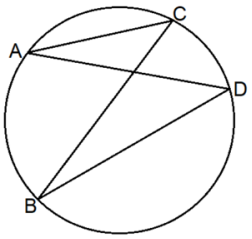
(a) $\frac{1}{4}$
(b) $\frac{3}{16}$
(c) $\frac{3}{8}$
(d) $\frac{4}{10}$
(e) $\frac{7}{16}$

31. Two circles of radius r overlap such that the outer edge of each circle passes through the center of the other circle. What is the area of their overlap?



(a) $A = (\frac{2\pi}{3} - \frac{\sqrt{3}}{2})r^2$
(b) $A = (\frac{3\pi}{2} - \frac{\sqrt{3}}{2})r^2$
(c) $A = (\frac{2\pi}{3} - \frac{1}{2})r^2$
(d) $A = (\frac{3\pi - 1}{2})r^2$
(e) None of the above

32. In the figure below, $\angle ACB$ and $\angle ADB$ are inscribed in the circle. Which of the following statements is true?



(a) The measure of $\angle ACB$ is greater than the measure of $\angle ADB$.
(b) The measure of $\angle ACB$ is less than the measure of $\angle ADB$.
(c) The measure of $\angle ACB$ is equal to the measure of $\angle ADB$.
(d) There is no relationship between the measure of $\angle ACB$ and the measure of $\angle ADB$.
(e) There is not enough information to determine the relationship between the measure of $\angle ACB$ and the measure of $\angle ADB$.

33. Consider the equation $ax^2 + bx + c = 0$, where a, b, and c are constant with $a < 0$. Find the conditions on b and c such that the equation *always* has two real roots with opposite signs.

(a) $b = 0$, $c > 0$
(b) $b = 0$, $c = 0$
(c) $b = 0$, $c < 0$
(d) $b > 0$, $c > 0$
(e) Both (a) and (d)

34. How many real roots does the equation $9^{2x^2} - 9^{x^2} - 2 = 0$ have?

(a) Two
(b) Three
(c) None
(d) Four
(e) None of above

35. Given:

Set $A = \left\{\text{All values of } x \text{ such that } \frac{x^2-9}{x+2} \leq 0\right\}$
Set $B = \left\{\text{All values of } x \text{ such that } 3x^2(x+3)(x^2-6x+9) \leq 0\right\}$
Set $C = \left\{\text{All values of } x \text{ such that } \frac{(x-3)(x-2)^3}{x+3} \leq 0\right\}$

Find: $A \cap B \cap C$

(a) $(-\infty, -3]$
(b) $(-\infty, -3)$
(c) $(-\infty, -3) \cup [2, 3]$
(d) $(-\infty, -3] \cup (-2, 3]$
(e) None of the above

36. How many solutions does $\sin x = 0.01x^2$ have?

(a) 3
(b) 4
(c) 5
(d) 6
(e) 7

37. A radiator has a capacity of C liters and is full with L percent antifreeze solution. How much of this should be drained and then replaced with H percent antifreeze solution to make a T percent antifreeze solution. (Note: $L < T < H$.)

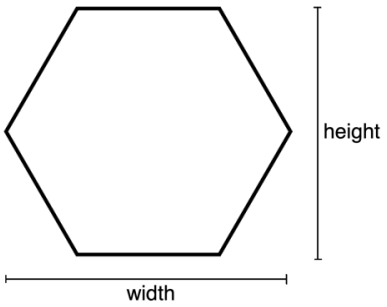
(a) $V = \frac{C(T-L)}{(H-L)}$
(b) $V = \frac{C(H-L)}{(T-L)}$
(c) $V = \frac{C(L-T)}{(L-H)}$
(d) $V = \frac{C(L-H)}{(L-T)}$
(e) None of the above

38. Find GCF(a, b) [that is, *the greatest common factor*], given:

$a = 1, 111, \ldots, 111$ (total of 100 ones in a row) and
$b = 11, 111, 111$.

(a) 1
(b) 11
(c) 101
(d) 1,111
(e) 10,001

39. In a regular hexagon, oriented like the one below, the height is increased by $66\frac{2}{3}\%$ and the width is decreased by 60%. What is the percent decrease in the area of the resulting hexagon?



(a) $33\frac{1}{3}\%$
(b) 25%
(c) 40%
(d) 50%
(e) 75%

40. A number N is a product of 10 different prime numbers ($N = p_1 \cdot p_2 \cdot ... \cdot p_{10}$). If M is the number of divisors strictly larger than 1, then:

(a) M is a perfect square
(b) M is odd
(c) M is even
(d) $M < 1000$
(e) None of the above