Name: ______________________________

Student ID: ___________________________

State Math Contest (Senior)

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are not allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked a), b), c), d), and e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- Scoring: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.
- You may not leave the room until at least 10:15 a.m.

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1. The front chain wheel of a bicycle has radius eight inches, and the rear chain wheel has radius four inches. The radius of the bicycle's wheels is 32 inches. If a cyclist pedals at the rate of three revolutions per second, how fast is she going?

   a) 8π feet/second b) $128\pi^2$ feet/second c) 32π feet/second

   d) 64 feet/second e) 96 feet/second

   Correct answer: c) 32π feet/second.
   Solution: Since the chain does not stretch, the linear speed of the chain wheels is the same. Thus, $2\pi(8)(3) = 2\pi(4)(r)$, where r is the number of revolutions per second of the rear chain wheel. Since $r = 6$, it follows that the bike's speed is $2\pi(32)(6)$ inches/second, or 32π feet/second.

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2. A cold-water faucet can fill a sink in 12 minutes, and a hot-water faucet can fill the same sink in 15 minutes. The drain at the bottom of the sink can empty the sink in 25 minutes. If both faucets and the drain are open, how long will it take to fill the sink?

   a) $5\frac{15}{57}$ minutes b) $7\frac{3}{4}$ minutes c) $9\frac{1}{11}$ minutes

   d) $10\frac{4}{11}$ minutes e) $20\frac{4}{57}$ minutes

   Correct answer: c) $9\frac{1}{11}$ minutes.
   Solution: If both faucets are working together to fill the sink, their work rates are added together. However, the drain is working against the faucets so that work rate needs to be subtracted from the faucets work rate.

$$\frac{time}{cold\text{–}water\ rate} + \frac{time}{hot\text{–}water\ rate} - \frac{time}{drain\ rate} = 1$$
$$\frac{time}{12} + \frac{time}{15} - \frac{time}{25} = 1$$

   Solve the rational equation for the time needed to fill the sink by clearing out the denominators (LCD) and solving the equation. The resulting time to fill the sink is $9\frac{1}{11}$ minutes.

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3. Three friends go to a restaurant for lunch. At the end of the meal, each of the three friends flips a fair coin, getting either a head or a tail. If two of the three coin flip outcomes match, the person with the non-matching outcome pays for everyone's lunch. If all three coin flip outcomes are the same, each friend flips a fair coin again. What is the

probability that each friend has to flip a coin more than two times before it is decided which friend pays for lunch?

a) $\frac{1}{16}$ b) $\frac{1}{4}$ c) $\frac{1}{3}$

d) $\frac{9}{16}$ e) $\frac{3}{4}$

Correct answer: a) $\frac{1}{16}$.
Solution: For the first round, the decision is made if the three coins do not match, and that probability is $\frac{6}{8} = \frac{3}{4}$. The decision is made on the second round if the coin outcomes matched on the second round and did not match on the second round; that probability is $\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \left(\frac{3}{16}\right)$. The probability that the friends have to flip coins more than 2 times means the decision was not made on the first or second rounds; that probability is $1 - \frac{3}{4} - \frac{3}{16} = \frac{1}{16}$.

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4. Ryan, Eric, and Kim like to hike mountains. Kim hiked six mountains that neither Ryan nor Eric hiked. There were only four mountains that all three hikers climbed. There was only one mountain Eric and Ryan hiked that Kim did not hike. There were no mountains that only Eric and Kim hiked. Ryan hiked six times more mountains than Eric. Kim hiked $\frac{1}{3}$ of the mountains that Ryan hiked and two times more mountains that Eric hiked. How many mountains did only Ryan hike?

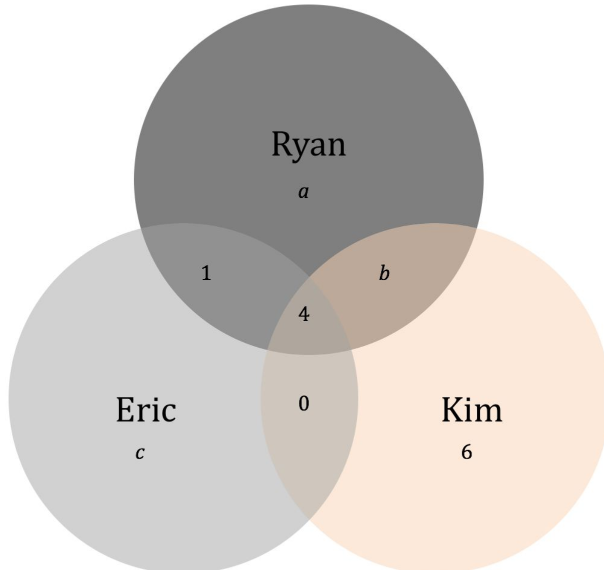
a) 22 b) 27 c) 29

d) 48 e) 56

Correct answer: c) 29.

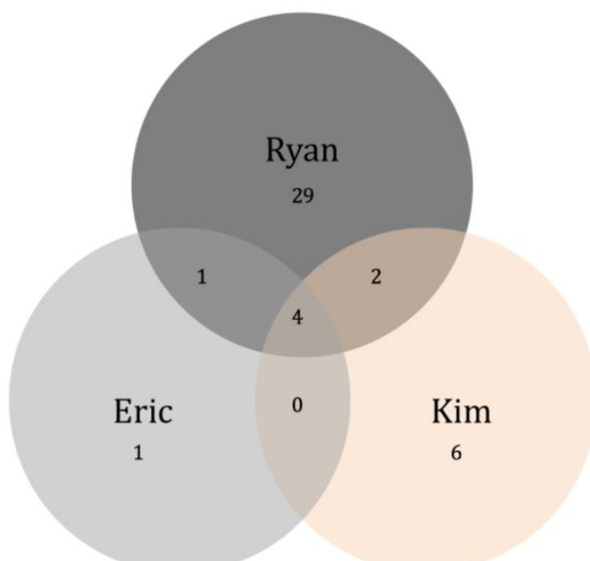
Solution: From the problem, we know that Ryan hiked the most mountains, then Kim, then Eric. Ryan hikes six times as much as Eric, three times as much as Kim, and Kim hikes twice as much as Eric. So, if we find how many mountains Eric hikes, we can multiply that number by two for Kim's amount and by six for Ryan's amount.

Using a Venn diagram, we know the following information and we use variables for the information we don't know:



If one were to use a system of equations to solve this problem, they would all be consistent and not helpful. A viable next step would be to use guess and check. Since Eric climbed the fewest mountains, let us assume he climbed just one by himself. Say, $c = 1$. Then, his total would be six mountains. Since Kim climbed twice as much as him, her total would then be 12. This would mean $b = 2$. If Ryan climbed three times as much as Kim, then his total is 36 (which is also six times as much as Eric) and, thus, a would have to be 29, which is answer option c.

Here is the Venn diagram of the solution:



Let's consider whether or not the other answer options are viable:
If we let $c = 2$, then Eric's total would be 7, Kim's total would be 14, and Ryan's total 42. Then, $b = 4$ and $a = 33$, which isn't an answer option.
If we let $c = 3$, then Eric's total would be 8, Kim's total would be 16, and Ryan's total 48. Then, $b = 6$ and $a = 37$, which isn't an answer option.
If we let $c = 4$, then Eric's total would be 9, Kim's total would be 18, and Ryan's total 54. Then, $b = 8$ and $a = 41$, which isn't an answer option.
If we let $c = 5$, then Eric's total would be 10, Kim's total would be 20, and Ryan's total 60. Then, $b = 10$ and $a = 45$, which isn't an answer option.
If we let $c = 6$, then Eric's total would be 11, Kim's total would be 22, and Ryan's total 66. Then, $b = 12$ and $a = 49$, which isn't an answer option.
If we let $c = 7$, then Eric's total would be 12, Kim's total would be 24, and Ryan's total 72. Then, $b = 14$ and $a = 53$, which isn't an answer option.
If we let $c = 8$, then Eric's total would be 13, Kim's total would be 26, and Ryan's total 78. Then, $b = 16$ and $a = 57$, which isn't an answer option.
We don't need to check higher values for c because no answer option is higher than 57.

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5. A rectangle is partitioned into 4 subrectangles as shown below. If the subrectangles have the indicated areas, find the area of the unknown rectangle.

210	240
91	?

a) 78 b) 98 c) 104

d) 270 e) 390

Correct answer: c) 104.
Solution:

$30 \times 7 = 210$	$30 \times 8 = 240$
$13 \times 7 = 91$	$13 \times 8 = 104$

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6. What is the coefficient of x^2y^4 when expanding the binomial $(2x + y)^6$?

a) 60 b) 30 c) 15

d) 120 e) 12

Correct answer: a) 60.
Solution: Use the binomial theorem which will tell you the coefficient is $2^2 \cdot ({}_6C_2) = 60$.

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7. Memphis is due north of New Orleans. Find the surface distance between Memphis (35° north latitude) and New Orleans (30° north latitude). Assume that the radius of Earth is 3960 miles.

a) 110π miles b) 19,800 miles c) 55π miles

d) $792\pi^2$ miles e) 55 miles

Correct answer: a) 110π miles.
Solution: The distance between New Orleans and Memphis is the arc length $s = r\theta$, where r is the radius of Earth, and θ is measured in radians. Now, $\theta = 5\left(\frac{2\pi}{360}\right) = \frac{\pi}{36}$.
Thus, $s = (3960)\left(\frac{\pi}{36}\right) = 110\pi$ miles.

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8. Two ships leave port at the same time, one heading due south and the other heading due east. Several hours later, they are 170 miles apart. If the ship traveling south traveled 70 miles farther than the other ship, how many miles did each ship travel?

a) Eastbound ship: 60 miles
Southbound ship: 130 miles

b) Eastbound ship: 65 miles
Southbound ship: 135 miles

c) Eastbound ship: 70 miles
Southbound ship: 140 miles

d) Eastbound ship: 75 miles
Southbound ship: 145 miles

e) Eastbound ship: 80 miles
Southbound ship: 150 miles

Correct answer: e) Eastbound ship: 80 miles
Southbound ship: 150 miles
Solution: The two ships are traveling perpendicular to each other, creating a right triangle. If the distance between the two ships is 170 miles, that must be the hypotenuse

of the right triangle. If the Eastbound ship's distance is set to x, the Southbound ship's distance needs to be set to
(x + 70). With all three sides of the right triangle determined (distances between the port and the ships and the distance between the ships themselves), the Pythagorean Theorem may be used.

$$(\textit{Southbound distance})^2 + (\textit{Eastbound distance})^2 = (\textit{Distance between ships})^2$$

$$(x+70)^2 + x^2 = (170)^2$$
$$(x+70)(x+70) + x^2 = 28,900$$
$$x^2 + 140x + 4900 + x^2 = 28,900$$
$$2x^2 + 140x + 4900 = 28,900$$
$$2x^2 + 140x - 24,000 = 0$$
$$2(x^2 + 70x - 12,000) = 0$$
$$2(x+150)(x-80) = 0$$

The resulting answers for x would $x = -150$ miles and $x = 80$ miles. Since distance cannot be negative, the only possible value of x would be $x = 80$ miles. Therefore, the Eastbound ship's distance would be $x = 80$ miles and the Southbound ship's distance would be $(x + 70) = 150$ miles.

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9. What is the difference between the sum of the first 500 positive even numbers and the first 500 positive odd numbers?

a) 5 b) 10 c) 100

d) 500 e) 1000

Correct answer: d) 500.
Solution: (2 + 4 + 6 + 8 + 10 + … + 1000) – (1 + 3 + 5 + 7 + 9 + … + 999) = (2 – 1) + (4 – 3) + (6 – 5) + (8 – 7) + (10 – 9) + … + (1000 – 999) = 1 + 1 + 1 + 1 + 1 + … + 1 = 500.

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10. Jamie has 100 feet of fencing material to enclose a rectangular exercise run for her dog. One side of the run will border her house, so she will only need to fence off the other three sides. What dimensions will give the enclosure the maximum area?

a) 20 ft × 20 ft b) 20 ft × 50 ft c) 25 ft × 25 ft

d) 25 ft × 50 ft   e) 50 ft × 50 ft

Correct answer: d) 25 ft × 50 ft

Solution: Since one side of the exercise run is the house, the 100 feet of fencing only needs to cover 2 widths and 1 length of the rectangular shape. (Let x be the width of the rectangle.)

$100 = x + x +$ length, then solving for length we get length $= 100 - 2x$

The area is given by the product of the width and length, so

Area = (width)(length)
$A(x) = (x)(100 - 2x)$
$A(x) = 100x - 2x^2$, then in quadratic form, $A(x) = -2x^2 + 100x$

To determine the maximum area, find the vertex of the quadratic equation.

$x_{symmetry} = \frac{-100}{2(-2)} = 25$
$A(25) = -2(25)^2 + 100(25) = 1250$

The maximum area of the rectangle will be 1250 ft^2 when the width of the rectangle is 25 feet. To find the length of the rectangle, divide the maximum area by the width.

length = 1250 ÷ 25
length = 50 feet

The dimensions of the enclosure should be 25 ft × 50 ft to produce the maximum area.

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11. Eratosthenes of Cyrene, Greece (276 - 194 BCE) was a scholar who lived and worked in Cyrene and Alexandria, Egypt. One day while visiting Syene, Egypt, he noticed that at noon the Sun's rays shone directly down a well. On this date one year later, in Alexandria, at noon he measured the angle of the Sun to be about 7.2 degrees. He estimated the distance between Alexandria and Syene to be 500 miles. Use this information to approximate the circumference of Earth.

a) $7{,}920\pi$ miles   b) 24,000 miles   (c) 133.3π miles

d) 25,000 miles

e) $\pi(3960)^2$ miles

Correct answer: d) 25,000 miles.

Solution: The distance between Syene and Alexandria is the arc length $s = r\boldsymbol{\theta}$ miles, where $s = 500$ miles and $\boldsymbol{\theta} = (7.2)\left(\frac{2\pi}{360}\right)$ radians. Thus, the radius of Earth is given by $500 = r(7.2)\left(\frac{2\pi}{360}\right)$, so that $r = \frac{(360)(500)}{2\pi(7.2)}$ $r = (360)(500)$ miles. This gives a circumference of $C = 2\pi r = 2\pi\left[\frac{(360)(500)}{2\pi(7.2)}\right] = 25{,}000$ miles.

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12. Let $z = -2 + 2i$ where $i^2 = -1$. Determine the value of z^4.

a) –64

b) $-16 + 16i$

c) 64

d) $16 + 16i$

e) $16 - 16i$

Correct answer: a) –64.

Solution:

$z = -2 + 2i$

$z^2 = (-2 + 2i)(-2 + 2i) = -8i$

$z^3 = z^2 \cdot z = (-8i)(-2 + 2i) = 16 + 16i$

$z^4 = z^3 \cdot z = (16 + 16i)(-2 + 2i) = -64$ OR $z^4 = z^2 \cdot z^2 = -8i \cdot -8i = -64$

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13. How many solutions does the equation

$$sin(x) + sin(2x) = 0$$

have in the interval $[0,\ 2\pi]$?

a) 5

b) 4

c) 3

d) 2

e) 0

Correct answer: a) 5.

Solution:

$sin(x) + sin(2x) = 0$

$sin(x) + 2sin(x)cos(x) = 0$

$sin(x)[1 + 2cos(x)] = 0$

Hence there are five solutions $x = 0,\ \pi,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi$.

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14. Convert 5432.1 in base 8 to base 10. What is the hundreds digit?

a) 2 b) 4 c) 0

d) 8 e) 1

Correct answer: d) 8.
Solution: 5432.1 in base 8 converts to 2842.125 in base 10. The answer is 8.

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15. It takes 867 digits to number the pages of a book. How many pages are there in the book?

a) 867 b) 434 c) 375

d) 325 e) 189

Correct answer: d) 325.
Solution: Pages 1-9 use 9 digits. Pages 10-99 (have 2 digits) use 180 digits. Pages 100-999 (3 digits) would use 900 digits. $867 - 189 = 678$ (3-digit numbers) pages. $678 \div 3 = 226$ pages. Thus, pages in the book: $99 + 226 = 325$ pages.

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16. Find the next number in the sequence $\frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{10}{11}, \ldots$

a) $\frac{11}{12}$ b) $\frac{16}{17}$ c) $\frac{12}{15}$

d) $\frac{12}{13}$ e) None of the above.

Correct answer: d) $\frac{12}{13}$
Solution: The pattern is $\frac{prime-1}{prime}$. The next prime after 11 is 13.

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17. If the number 18! is divided by 19, what will be the remainder?

a) 1 b) 3 c) 6

d) 12 e) 18

Correct answer: e) 18.
Solution: The most concrete approach is to experiment and look for a pattern. If $(n-1)!$ is divided by n, the remainder is always either 0 (if n is composite) or $n-1$ (if n is prime). For instance,

n	2	3	4	5	6	7	8
Remainder of $(n-1)!$ divided by n	1	2	0	4	0	6	0

Thus the solution here is 18 = 19 – 1.

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18. There are six novels that you want to read over the summer. Two of the novels are by the same author and you don't want to read those directly after one another. In how many orders can you read the six novels?

a) 36 b) 240 c) 480

d) 600 e) 720

Correct answer: c) 480.
Solution: There are 6! = 720 orders in which to read the six novels if there were no restrictions. The ways that are not acceptable are when the two novels by the same author are read directly after one another. The number of orders that are not acceptable can be calculated by letting the two novels by the same author be one item, then you arrange 5 items (the combined pair of novels by the same author and the other 4 novels) in 5! ways, and then there are 2 orders in which to read the two novels by the same author, for a total of 5! × 2 = 240 orders that are not acceptable. There are
720 – 240 = 480 orders that are acceptable.

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19. A fair coin is tossed 4 times. What is the probability that exactly one run of 2 straight heads occurs?

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{2}$

d) $\frac{5}{16}$ e) $\frac{7}{16}$

Correct answer: d) $\frac{5}{16}$.
Solution: There are 16 ways to toss 4 coins, and 5 ways to get exactly one run of 2 heads, namely HHTT, HHTH, THHT, TTHH, HTHH. Any longer run of heads would give more than one run of 2 straight heads.

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20. What is the area of a triangle with sides of length 5, 5, and 8?

a) 6 b) 12 c) 20

d) 25 e) 40

Correct answer: b) 12.
Solution: The isosceles triangle is made up of 2 congruent right triangles of hypotenuse length 5 and leg lengths 4 and 3 by the Pythagorean Theorem. So the area is $2 \cdot \left(4 \cdot \frac{3}{2}\right) = 12$.

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21. Find all the values of x which satisfy the inequality $\sqrt{(x-2)^2} < |x|$.

a) $(-2, 0)$ b) $(-2,\ 1) \cup (2,\ \infty)$ c) $(1, \infty)$

d) $(2, 5)$ e) None of the above

Correct answer: c) $(1, \infty)$.
Solution: With $|x-2| < |x|$, we have three possible cases. Case 1, $x < 0$, no solution. Case 2, $0 \le x \le 2$, solution is (1, 2]. Case 3, $x > 0$, solution is $(2,\ \infty)$.

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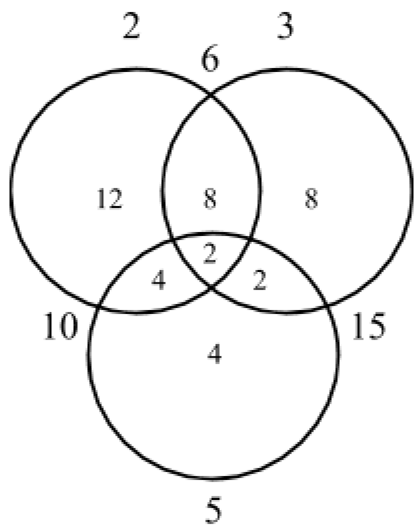
22. How many positive integers less than or equal to 60 have either 2, 3, or 5 as a factor?

a) 12 b) 30 c) 44

d) 62 e) 72

Correct answer: c) 44.
Solution:
$60 \div 30 = 2$
$60 \div 15 = 4$
$60 \div 10 = 6$
$60 \div 6 = 10$
$60 \div 2 = 30$
$60 \div 3 = 20$
$60 \div 5 = 12$



44 positive integers have factors of 2, 3, or 5.

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23. Solve the equation $\sqrt[3]{7^{x-1}} = \sqrt{7^{2x+1}}$.

a) $x = \frac{5}{4}$ b) $x = -\frac{5}{4}$ c) $x = \frac{4}{5}$

d) $x = -\frac{4}{5}$ e) None of the above.

Correct answer: b) $x = -\frac{5}{4}$.
Solution: $\sqrt[3]{7^{x-1}} = \sqrt{7^{2x+1}} \to 7^{\frac{x-1}{3}} = 7^{\frac{2x+1}{2}} \to \frac{x-1}{3} = \frac{2x+1}{2} \to 2x - 2 = 6x + 3 \to 4x = -5 \to$ $x = -\frac{5}{4}$.

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24. If the quadratic function $y = x^2 + ax + |a|$ has two distinct real zeros, then the interval for the value a is

a) $(-\infty, 0) \cup (4, \infty)$ b) $(-\infty, -4) \cup (4, \infty)$ c) $(-4, 4)$

d) $(-\infty, \infty)$ e) $(0, \infty)$

Correct answer: b) $(-\infty, -4) \cup (4, \infty)$.
Solution: Let $\Delta = a^2 - 4|a| = a(a-4)$,
if $a > 0$ or $a(a+4)$, if $a < 0$.

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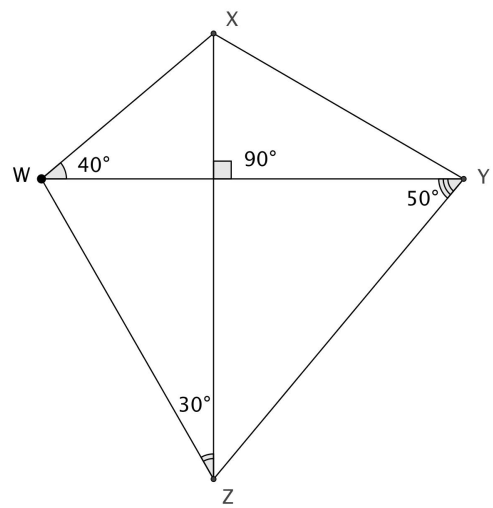
25. What is the measure of $\angle ZXY$ in the given figure?



a) Impossible to determine b) 45° c) 50°

d) 55° e) 60°

Correct answer: e) 60°.
Solution: Let A be the point where the two segments cross and form four right angles. Then $\triangle WXA$ is similar to $\triangle ZYA$. This, in turn, forces $\triangle XYA$ to be similar to $\triangle WZA$ by SAS.

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26. Seven friends took a quiz. Each got a score that is a whole number between 1 and 100. No two friends got the same score. The median score received by the friends was 50 and the range (the maximum score minus the minimum score) was 20. What is the highest score that any of the friends could have received?

a) 50 b) 53 c) 60

d) 67 e) 99

Correct answer: d) 67.
Solution: The 4th highest score must be 50, the median. Since the range is fixed at 20, to maximize the highest score, the lowest score must be maximized. The highest values that the three lowest performing friends could get would be 49, 48, 47. With the lowest value at 47, the highest value must be 67.

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27. Find the last digit of 3^{999}.

a) 1 b) 3 c) 7

d) 9 e) None of the above.

Correct answer: c) 7.

Solution: Look for a pattern:

$3^1 = \underline{3}$

$3^2 = \underline{9}$

$3^3 = 2\underline{7}$

$3^4 = 8\underline{1}$

$3^5 = 24\underline{3}$

$3^6 = 72\underline{9}$

$3^7 = 218\underline{7}$

$3^8 = 656\underline{1}$

The pattern repeats every four numbers: 3, 9, 7, 1. So, 999 ÷ 4 = 249 R3. Thus, the last digit is 7.

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28. Let a, b, c, and d be positive integers and let $log_a d = 24$. What is numerical value of $log_c \frac{1}{d} \cdot log_{a^{1.2}} \sqrt{b} \cdot log_{b^{-2}} c^5$?

a) $-\frac{3}{2}$ b) 1 c) $\frac{3}{2}$

d) 24 e) 25

Correct answer: e) 25.

Solution: Observe that $log_c \frac{1}{d} = -log_c d$, $log_{a^{1.2}} \sqrt{b} = \frac{0.5}{1.2} log_a b$, and $log_{b^{-2}} c^5 = -\frac{5}{2} log_b c$.

Next, $log_a b \cdot log_b c = log_a b^{log_b c} = log_a c$ and $log_a c \cdot log_c d = log_a c^{log_c d} = log_a d$.

Hence, $log_c \frac{1}{d} \cdot log_{a^{1.2}} \sqrt{b} \cdot log_{b^{-2}} c^5 = (-1)\left(\frac{0.5}{1.2}\right)\left(-\frac{5}{2}\right) log_a d = (-1)\left(\frac{0.5}{1.2}\right)\left(-\frac{5}{2}\right) 24 = 25.$

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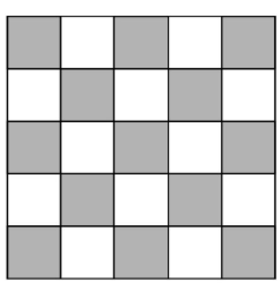
29. How many rectangles can be made using the squares on a checkerboard that is 5 squares by 5 squares?



a) 64 b) 81 c) 256

d) 625 e) 225

Correct answer: e) 225.

Solution: A rectangle is formed by choosing left and right sides (from 6 available places) and top and bottom sides (from 6 available places). Thus the solution is $\binom{6}{2} \times \binom{6}{2}$ $= 15 \times 15 = 225$ (where $\binom{6}{2}$ is the number of ways to choose two items from six available, without regard to order).

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30. Find the slope of the line through the point $P = (1, 1)$ on the parabola $y = x^2$ which does not intersect this parabola at any other point.

a) 0 b) 1 c) 2

d) –1 e) None of the above

Correct answer: c) 2.

Solution: The equation of the line that intersects the given parabola at the point where $x = 1$ and $y = 1$ is $y = m(x - 1) + 1$. Since this line is supposed to intersect the parabola only at one point it follows that the equation $x^2 = m(x - 1) + 1$ has a unique solution only for $m = 2$.

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31. Solve for x:

$$x - 4 - \frac{3x}{x+4} = \frac{-1}{1 - \frac{1}{1+\frac{1}{2}}}$$

a) –8 b) –2 and 2 c) –4 and 4

d) $\frac{3 \pm \sqrt{61}}{2}$ e) There are no real solutions.

Correct answer: b) –2 and 2.

Solution: The equation eventually simplifies to $(x + 4)\left(x - 4 - \frac{3x}{x+4}\right) = -3(x + 4)$. This simplifies to $x^2 - 16 - 3x = -3x - 12$ which goes to $x^2 - 16 - 3x + 3x + 12 = 0$ and this simplifies to $x^2 - 4 = (x + 2)(x - 2) = 0$. This x is 2 or –2.

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32. Let n be a positive integer and define a function f such that $f(n) =$ the product of the digits of n. Which of the following cannot be a value of f?

a) 0 b) 1 c) 99

d) 100 e) 10^{100}

Correct answer: c) 99.
Solution: Since 99 has a prime factor of 11, which is not a digit. The rest have either 0s, 1s, or primes in their factorizations that are 2 or 5, making them possible.

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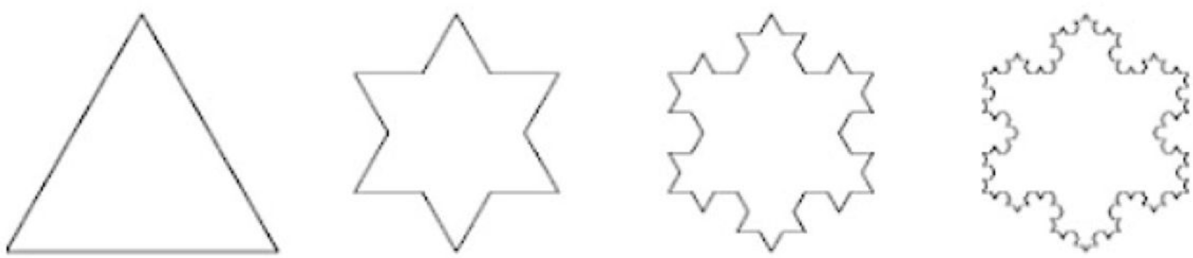
33. The image below shows the first four stages of creating a Koch Snowflake. It is formed by beginning with an equilateral triangle and then adding to each side of that triangle another triangle one ninth of the size of the previously added triangle. The table below shows how the area of the Koch Snowflake grows in a geometric sequence pattern. Note the table uses an initial side length of 1 unit.



Stage	Area
1	$\frac{\sqrt{3}}{4}$
2	$\frac{\sqrt{3}}{4}\left(1 + 3 \cdot \frac{1}{9}\right)$
3	$\frac{\sqrt{3}}{4}\left(1 + \frac{3}{9} + \frac{3 \cdot 4}{9^2}\right)$
4	$\frac{\sqrt{3}}{4}\left(1 + \frac{3}{9} + \frac{3 \cdot 4}{9^2} + \frac{3 \cdot 4^2}{9^3}\right)$
n	$\frac{\sqrt{3}}{4}\left(1 + \sum_{n=1}^{\infty} \frac{3}{9} \cdot \frac{4^{n-1}}{9^{n-1}}\right)$

What is the limit of the area of the Koch Snowflake shown in the table?

a) $\frac{\sqrt{3}}{4}$ b) $\frac{3\sqrt{3}+4}{12}$ c) $\frac{2\sqrt{3}}{5}$

d) $\frac{\sqrt{3}}{3}$ e) ∞

Correct answer: c) $\frac{2\sqrt{3}}{5}$.

Solution:

$$\frac{\sqrt{3}}{4}\left(1+\sum_{n=1}^{\infty}\frac{3}{9}\cdot\frac{4^{n-1}}{9^{n-1}}\right)=\frac{\sqrt{3}}{4}\left(1+\left(\frac{\frac{1}{3}}{1-\frac{4}{9}}\right)\right)=\frac{\sqrt{3}}{4}\left(1+\frac{1}{3}\cdot\frac{9}{5}\right)=\frac{\sqrt{3}}{4}\left(\frac{8}{5}\right)=\frac{2\sqrt{3}}{5}$$

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34. The equation for a particular parabola is given by $y=\frac{x^2-6x+9}{-12}+1$. A certain circle has the following properties: the center of the circle is at the focus of that parabola, and the circle intersects that parabola's vertex. Find the equation of the circle.

a) $(x-2)^2+(y-3)^2=9$ b) $(x-2)^2+(y+3)^2=3$

c) $(x-3)^2+(y+2)^2=3$ d) $(x-3)^2+(y+2)^2=9$

e) $(x-3)^2+(y-2)^2=9$

Correct answer: d) $(x-3)^2+(y+2)^2=9$.

Solution: The equation for the parabola can be rewritten to yield $(y-1)=\frac{(x-3)^2}{-12}$. In vertex form, we see that the vertex is (3, 1). The distance from the vertex to the focus is given by $\left|\frac{-12}{4}\right|=3$. Since the parabola points down, this means the focus, and thus the center of the circle, is at (3, –2). Finally, we recognize that for the circle to intersect the vertex, it must be that the radius of the circle is also 3, yielding the equation for the circle $(x-3)^2+(y+2)^2=9$.

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35. Consider a polynomial $p(x)$ with integer coefficients. Let us assume that there are three distinct integers a_1, a_2, and a_3 such that $p(a_1)=p(a_2)=p(a_3)=1$. At most how many integer roots does $p(x)$ have?

a) 0 b) 1 c) 2

d) 3 e) 5

Correct answer: a) 0.

Solution: Let us observe that for any integers a and b, and a positive integer k:

$$a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + ... + ab^{k-2} + b^{k-1}\right).$$

Hence, for the polynomial $p(x)$ with integer coefficients, $a-b$ divides $p(a) - p(b)$. Next, let us assume that an integer number b is a root of $p(x)$. Then $p(b) = 0$ and $p(a_1) - p(b) = p(a_2) - p(b) = p(a_3) - p(b) = 1$.

Hence $a_1 - b$, $a_2 - b$, and $a_3 - b$ all divide 1. This implies that the integers $a_1 - b$, $a_2 - b$, and $a_3 - b$ must equal to ± 1. So, only two out of three could be distinct and at least one pair, say $a_1 - b$ and $a_2 - b$, must equal.

However, $a_1 - b = a_2 - b$ implies that $a_1 = a_2$, which contradicts a condition of the problem that the integers a_1, a_2, and a_3 are all distinct.

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36. Consider the line of equation $y = a - x$ for $a > 0$. The only value for the constant $a > 0$ that will make the line tangent to the circle of equation $x^2 + y^2 = 1$ is:

a) 1 b) $\sqrt{2}$ c) 2

d) 3 e) None of the above

Correct answer: b) $\sqrt{2}$.
Solution: The given line has slope –1. Because we want the line to be tangent to the circle we need it to be perpendicular to the radius at the point where the line is tangent. It follows that the slope of the radius has to be 1. The only radius of the given circle that has radius 1 is the one along the line $x = y$. Therefore, at the point where the line is tangent to the circle both x and y will equal $\frac{\sqrt{2}}{2}$. Since we want this point to be on the line $y = a - x$, it follows that $\frac{\sqrt{2}}{2} = a - \frac{\sqrt{2}}{2}$ and hence $a = \sqrt{2}$.

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37. $(3\sqrt{7}+4)^3 + (3\sqrt{7}-4)^3 =$

a) 1 b) $3\sqrt{7}$ c) $21\sqrt{7}$

d) $111\sqrt{7}$ e) $666\sqrt{7}$

Correct answer: e) $666\sqrt{7}$

Solution: Let $a = 3\sqrt{7}+4$ and $b = 3\sqrt{7}-4$.
Then we have $a^3 + b^3$. Factor sum of cubes to be $(a+b)\left(a^2 - ab + b^2\right)$.
Find $a+b$ to be $\left(3\sqrt{7}+4\right)+\left(3\sqrt{7}-4\right) = 6\sqrt{7}$.
Find a^2 to be $\left(3\sqrt{7}+4\right)\left(3\sqrt{7}+4\right) = 79 + 24\sqrt{7}$.
Find ab to be $\left(3\sqrt{7}+4\right)\left(3\sqrt{7}-4\right) = 47$.
Find b^2 to be $\left(3\sqrt{7}-4\right)\left(3\sqrt{7}-4\right) = 79 - 24\sqrt{7}$.
Then,

$$\begin{aligned}(a+b)\left(a^2-ab+b^2\right) &= \left(6\sqrt{7}\right)\left[\left(79+24\sqrt{7}\right)-(47)+\left(79-24\sqrt{7}\right)\right]\\ &= 6\sqrt{7}\left(79+24\sqrt{7}-47+79-24\sqrt{7}\right)\\ &= 6\sqrt{7}\,(158-47)\\ &= 6\sqrt{7}\cdot 111\\ &= 666\sqrt{7}\end{aligned}$$

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38. A 300- room hotel is two thirds filled when the nightly room rate is \$90. For each \$5 increase in cost results in 10 fewer occupied rooms. Find the nightly rate that will maximize income.

a) \$100 b) \$85 c) \$110

d) \$90 e) \$95

Correct answer: e) \$95.
Solution: Let x denote the number of \$5 increments in the room rate. For example, if x = 3 it simply means that we decided to increase the rate by \$15 from the original price of \$90. Moreover, if $x = -1$ it means that we cut the price by \$5. Let us assume that we decide to apply x increments of \$5. Then the price will be \$$(90 + 5x)$ and the number of occupied rooms will be $200 - 10x$. It follows that the income will be I = \$$(90 + 5x)(200 - 10x)$. Please note that the equation for the income is quadratic with the quadratic term negative. This means that the peak income is at the vertex of this parabola. Looking at the equation of the parabola as is factored we see that the x-intercepts (where the income is zero) are $x = -18$ and $x = 20$. Since the vertex is halfway between the two x-intercepts, it follows that $x = 1$ is the x of the vertex and hence the x value that will give the peak value for the income. Therefore, the nightly rate that will maximize the income is given by e) \$95.

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39. If $f(x-1) = (1-x)(x+2)(x-3)$, then one of the antiderivatives of $f(x+1)$ is

a) $\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{x^2}{2} - 4x$ b) $-x^4 - 4x^3 + x^2 + 4$

c) $-\frac{1}{8}\left(x^6+8x^5+14x^4-16x^3-31x^2+8x+16\right)$ d) $\frac{1}{2}(x+1)^2$

e) None of the above

Correct answer: e) None of the above.
Solution:
$f(x+1)=f\left((x+2)-1\right)=\left(1-(x+2)\right)\left((x+2)+2\right)\left((x+2)-3\right)=-(x+1)(x+4)(x-1)$

$\int f(x+1)\ dx=-\int(x+1)\ dx=-\frac{x^4}{4}-\frac{4}{3}x^3+\frac{x^2}{2}+4x$

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40. What is the minimum period of the function $f(x)=\frac{cos(x)+1}{cot\left(\frac{x}{2}\right)}-sin^2(x)\cdot tan\left(\frac{x}{2}\right)$?

a) $\frac{\pi}{2}$ b) π c) 2π

d) 3π e) 4π

Correct answer: b) π.
Solution:
$\frac{cos(x)+1}{cot\left(\frac{x}{2}\right)}-sin^2(x)\cdot tan\left(\frac{x}{2}\right)$
$=\left[cos(x)+1-sin^2(x)\right]tan\frac{x}{2}=\left[cos(x)+cos^2(x)\right]tan\frac{x}{2}$
$=cos(x)\left[1+cos(x)\right]tan\frac{x}{2}=cos(x)\cdot 2cos^2\left(\frac{x}{2}\right)\cdot tan\left(\frac{x}{2}\right)$
$=cos(x)\cdot 2cos^2\left(\frac{x}{2}\right)\cdot\left[\frac{sin\left(\frac{x}{2}\right)}{cos\left(\frac{x}{2}\right)}\right]=2\cdot cos(x)\cdot sin\left(\frac{x}{2}\right)\cdot cos\left(\frac{x}{2}\right)$
$=cos(x)\cdot sin(x)=\frac{1}{2}sin(2x)$.
