Name:________________________________

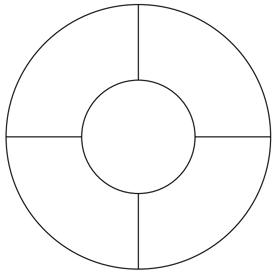
Student ID:______________________________

State Math Contest (Junior)

Instructions:

- Do not turn this page until your proctor tells you.
- Enter your name, grade, and school information following the instructions given by your proctor.
- Calculators are **not** allowed on this exam.
- This is a multiple choice test with 40 questions. Each question is followed by answers marked a), b), c), d), and e). Only one answer is correct.
- Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
- **Scoring:** You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- You will have 2 hours and 30 minutes to finish the test.

1. In the figure below we want to color the five regions. If we have to color the regions so that it does not have the same color as an adjacent region, what is the fewest number of colors we can use?

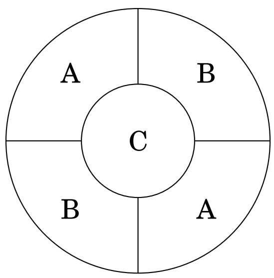


a) 1 b) 2 c) 3

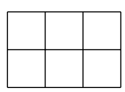
d) 4 e) 5

Solution:

Correct answer: c



2. A young man has to walk from his home to his work. (His mom picks him up so he does not walk home.) How many days can he walk to work in a different way by walking a total of 2 blocks north and 3 blocks east in any order? (No cutting diagonal through the blocks).



a) 1 b) 5 c) 8

d) 10 e) 12

Solution:

Correct answer: d
The young man needs to walk a total of 5 blocks. 2 of the blocks need to be north and the remaining east. ${}_5C_2 = 10$

3. The current in a stream moves at a constant speed. A duck swims 8 miles downstream in 2 hours. She returns upstream in 14 hours. How fast does the duck swim in still water? (You should assume that the duck swims the same speed relative to the water both directions.)

a) 15/7 miles per hour. b) 14/7 miles per hour. c) 16/7 miles per hour.

d) 18/7 miles per hour. e) 13/7 miles per hour.

Solution:

Correct answer: 16/7 miles per hour.
The current helps the duck swim downstream and hinders her swimming upstream. Let the duck's speed in still water be x and the speed of the current be y. Then, swimming downstream the duck's speed is $x+y$, while swimming upstream it is $x-y$. Hence, we have for downstream

$$(x+y)2 = 8$$

and for upstream

$$(x-y)14 = 8.$$

Solving these equations, we have $x = 16/7$ and $y = 12/7$. Hence, the duck's speed in still water is 16/7 miles per hour.

4. You are rolling 2 dice. What is the probability that the absolute value of the difference of the outcomes is at least 4?

a) $\frac{1}{6}$ b) $\frac{1}{2}$ c) $\frac{1}{5}$

d) $\frac{1}{9}$ e) $\frac{3}{5}$

Solution:

Correct answer: a
There are 6 options for each die making the total number of out comes $6 \times 6 = 36$. The ways you could get a difference of more than 3 are:

$$1,6;, \quad 6,1; \quad 1,5; \quad 5,1; \quad 2,6; \quad 6,2.$$

Making the probability $\frac{6}{36} = \frac{1}{6}$.

5. A triangle in the xy-plane has vertices at $(7,3)$, $(12,3)$, and $(c+7,15)$. Find values of c so that this is a right triangle.

a) 4 b) 3 and −2 c) 8 and −4

d) 0 and −3 e) 0 and 5 f) 9

Solution:

Correct answer: e
For the three points to form a right triangle $c+7$ needs to equal 7 or 12. Thus $c = 0,5$.

6. A parabola in the xy-plane is known to have its vertex at $(2,5)$ and its focus 2 units to the left of the vertex. What is its equation?

a) $(y-2)^2 = -8(x-5)$ b) $(y-5)^2 = -8(x-2)$ c) $(y-5)^2 = 2(x-2)$

d) $(y-2)^2 = -2(x-5)$ e) $(y-5) = 4(x-2)^2$

Solution:

Correct answer: b

7. A bumble bee is traveling back and forth between the front end of two trains moving towards each other. If the trains start 90 miles away from each other and one train is going 10 miles per hour while the other train is going 20 miles an hour and the bumble bee is traveling 100 miles per hour, how many miles does the bumble bee travel before being smashed by the two trains colliding?

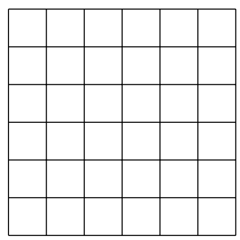
a) 450 miles   b) 90 miles   c) 300 miles

d) 180 miles   e) 270 miles

Solution:

Correct answer: c
The trains start 90 miles apart and collectively are going 30 mile per hour. It will take 3 hours for the trains to collide. Since the bubble bee is traveling 100 miles per hour and travels for 3 hours, the bee travels 300 miles.

8. How many ways are there to place 6 circles in 6 different squares on the board below so that no circle is in the same row or column as another circle?



a) 360   b) 720   c) 120

d) 12   e) 36

Solution:

Correct answer: b
We can place a circle row by row.
First Row: 6 choices
Second Row: 5 choices left
Third Row: 4 Choices left
Forth Row: 3 Choices left
Fifth Row: 2 Choices left
Sixth Row: 1 Choice left

$$(6)(5)(4)(3)(2)(1) = 6! = 720$$

9. Find the area of triangle ΔABC if $AB = AC = 50$ in and $BC = 60$ in.

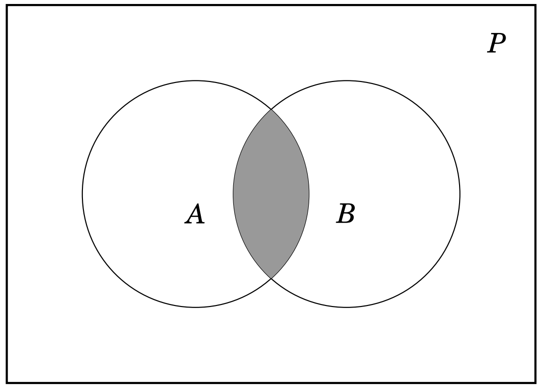
a) 2000 square inches   b) 1500 square inches   c) 1000 square inches

d) 2400 square inches   e) 1200 square inches

Solution:

Correct answer: e
Since the triangle is isosceles , the altitude AX to the side BC bisect BC. Hence, using the Pythagorean Theorem on the right triangle ABX, we have $AX = 40$ in. Therefore, the area is $(BC)(AX)/2 = 60 \times 40/2$ square inches $= 1200$ square inches.

10. Which of the following expressions indicate the shaded region. ($\overline{A}$ represents the complement of A relative to P.)



I. $A \cup B$ II. $A \cap B$ III. $P \cup (A \cup B)$ IV. $P \cap (A \cup B)$
V. $P \cap (A \cap B)$ VI. $P \cup (A \cap B)$ VII. $\overline{\overline{A} \cup \overline{B}}$ VIII. $\overline{\overline{A} \cap \overline{B}}$

a) only II.
b) only V.
c) only VIII.
d) II., V., and VII.
e) III., V., and VIII.

Solution:

Correct answer: d

11. In the sequence of numbers $1, 4, 3, -1, \cdots$ each term after the first two is equal to the term preceding it minus the term preceding that. Find the sum of the first one hundred terms of the sequence.

a) 1
b) -2
c) -3
d) 7
e) 3

Solution:

Correct answer: d
The sequence is periodic with period 6. The first 6 terms are 1, 4, 3, -1, -4, -3 The sum of any six consecutive terms is zero. Since 96 is a multiple of six, the sum of the first 96 terms is zero and the sum of the first hundred terms is the same as the sum of the first 4 terms. 1 + 4 + 3 + (-1) = 7.

12. What is the smallest positive integer that is both a perfect power of 11 ($11^1, 11^2, 11^3 \ldots$) and **not** a palindrome? (A palindrome is a number that reads the same backwards as forwards such as 121 or 1331)

a) 1771561
b) 161151
c) 14631
d) 161051
e) 122

Solution:

Correct answer: d
$11^1 = 11$, $11^2 = 121$, $11^3 = 1331$, $11^4 = 14641$, $11^5 = 161051$

13. A positive whole number leaves a remainder of 7 when divided by 11 and a remainder of 10 when divided by 12. What is the remainder when divided by 66?

a) 0
b) 40
c) 28
d) 52
e) 29

Solution:

Correct answer: b

$$n = 11a + 7 = 12b + 10$$

Looking at $11a + 7$ we have numbers

$$7, 18, 29, 40, 51, 62, 73, 84, 95, 106, 117, \ldots$$

Looking at $12b + 10$ we have numbers

$$10, 22, 34, 46, 58, 70, 82, 94, 106, 118, \ldots$$

$11a + 7 = 12b + 10$ at 106

$$106 = 66(1) + 40$$

Thus, the remainder is 40.

14. Suppose M, N, O, P are real numbers. Suppose that the following two conditions hold.

- M is greater than both N and O
- N is greater than O and less than P

Which of the following statements must be true?

a) M is not greater than either O and P.
b) O is greater than N.
c) P is greater than O.
d) O is greater than P.
e) M is greater than P, O, and N.

Solution:

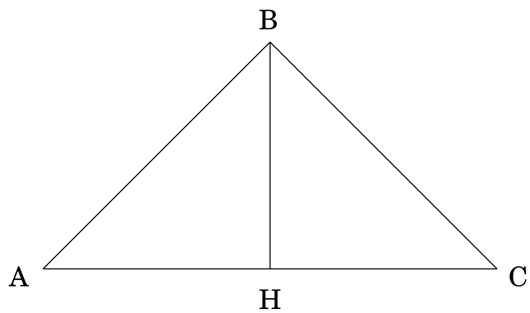
Correct answer: c
We have the following situations:

(i) M = P > N > O
(ii) M > P > N > O
(iii) P > M > N > O

The only one of the above statements that is true in each case is:

P is greater than O.

15. Line BH is perpendicular to line AC. Angle BAC is equal to BCA with both measure $\frac{\pi}{6}$. If BH is length 4, what is the length of AC?
Diagram is may not be drawn to scale.



a) $8\sqrt{3}$ b) $4\frac{\sqrt{3}}{2}$ c) 16

d) 10 e) $4\frac{\sqrt{2}}{2}$

Solution:

Correct answer: a
The triangle $\triangle ABH$ is congruent to the triangle $\triangle BCH$.
The angle $\angle BAH$ is complementary to the angle $\angle ABH$ since $\triangle ABH$ is a right triangle.
Thus $\triangle ABH$ is a $\left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right)$ right triangle and AH has length $4\sqrt{3}$. So the length of AC is $8\sqrt{3}$.

16. Find the sum of all the even integers from 546 to 854 inclusive:

$$546 + 548 + 550 + \cdots + 852 + 854.$$

a) 106,400 b) 109,200 c) 107,800

d) 107,100 e) 108,500

Solution:

Correct answer: e

$$(546+854)+(548+852)+(550+850)+\cdots+(696+704)+(698+702)+700$$

$$1400+1400+1400+\cdots+1400+1400+700$$

Since $698-546 = 152$ and we have only even numbers we have $76+1 = 77$ times 1400 are added together.

$$(77 \times 1400) + 700 = 108500$$

17. Find a real number a such that equation $||x-a|-a| = 2$ has exactly three different solutions.

a) -1 b) 1 c) 0

d) -2 e) 2

Solution:

Correct answer: e

Removing one absolute value sign, we have

$$|x-a| = a \pm 2.$$

It is clear that when $a = 2$, there are three solutions: $2, 6, -2$.

18. Let x and y be positive numbers satisfying

$$2 + \log_2 x = 3 + \log_3 y = \log_6(x+y).$$

Find the value of $\frac{1}{x} + \frac{1}{y}$.

a) 54 b) 36 c) 108

d) 216 e) 81

Solution:

Correct answer: c

Let

$$a = 2 + \log_2 x = 3 + \log_3 y = \log_6(x+y).$$

Then,

$$x = 2^{a-2},\ y = 3^{a-3},\ (x+y) = 6^a.$$

Hence,

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{6^a}{2^{a-2}3^{a-3}} = 2^2 \times 3^3 = 108.$$

19. What is the minimal value of $4(x^2+y^2+z^2) - (x+y+z)^2$ when x, y, z are different integers?

a) 8 b) 9 c) 7

d) 6 e) 10

Solution:

Correct answer: a

Note

$$4(x^2+y^2+z^2) - (x+y+z)^2 = (x-y)^2 + (y-z)^2 + (z-x)^2 + x^2 + y^2 + z^2.$$

So it has the minimal value 8 when $x = 1$, $y = 0$, and $z = -1$.

20. The probability that a baseball player gets a hit is 1/5. Find the probability that the player gets exactly 2 hits when batting 4 times in his next game.

a) $\frac{95}{625}$ b) $\frac{98}{625}$ c) $\frac{92}{625}$

d) $\frac{94}{625}$ e) $\frac{96}{625}$

Solution:

Correct answer: e
The probability of getting a hit, a hit, a miss, and a miss, in the order is $\frac{1}{5}\frac{1}{5}\frac{4}{5}\frac{4}{5} = \frac{4^2}{5^4}$. We then multiply it by the number of possible order in which the hits can come, which is the number of ways to choose 2 of 4 at bats to be hits, that is 6. Thus, the total probability is $6 \times \frac{4^2}{5^4} = \frac{96}{625}$.

21. Find x if $3^{27^x} = 27^{3^x}$.

a) $x = 1/3$
b) $x = 1/2$
c) $x = 3/2$
d) $x = 1/4$
e) $x = 2/3$

Solution:

Correct answer: b
Express the both sides with the same base 3. Then we have

$$3^{27^x} = 27^{3^x} = (3^3)^{3^x} = 3^{(3)3^x} = 3^{3^{1+x}}.$$

Hence, $27^x = 3^{1+x}$. Again express the both sides with the same base 3, we have

$$3^{1+x} = (3^3)^x = 3^{3x}.$$

Hence $1 + x = 3x$, that is $x = 1/2$.

22. Find y if $(2, y)$ lies on the line joining $(0, 3/2)$ and $(9/4, 0)$.

a) $y = -1/6$
b) $y = 1/6$
c) $y = -1/3$
d) $y = 1/3$
e) $y = 5/6$

Solution:

Correct answer: b
The line in equation is given by $4x + 6y = 9$. Plugging in $x = 2$, we have $8 + 6y = 9$, hence $y = 1/6$.

23. Find the *shortest* path which starts at the origin and visits all five of the following points and returns to the origin: $\{(0,0), (1,0.5), (2,1), (2,0), (0,3)\}$.

a) $(0,0),(1,0.5),(2,1),(2,0),(0,3),(0,0)$
b) $(0,0),(2,0),(1,0.5),(2,1),(0,3),(0,0)$
c) $(0,0),(1,0.5),(2,0),(2,1),(0,3),(0,0)$
d) $(0,0),(0,3),(1,0.5),(2,0),(2,1),(0,0)$
e) $(0,0),(0,3),(2,0),(1,0.5),(2,1),(0,0)$

Solution:

Correct answer: c
By inspection the path (c) is shorter than the path (d) and the path (a) is shorter than the path (e). Let x be the distance from $(0,0)$ to $(1,0.5)$. Then the distance from $(1,0.5)$ to $(2,0)$ and the distance $(1,0.5)$ to $(2,1)$ are also both equal to x. Notice that $x > 1$. Let y be the $(2,0)$ to $(0,3)$ and z be the distance from $(2,1)$ to $(0,3)$. Then $z < y$.
The length of the path (a) is $2x + y + 4$.
The length of the path (b) is $2x + z + 5$.
The length of the path (c) is $2x + z + 4$.
Thus the path (c) is the shortest.

24. Find the volume of a cone whose base has an area of one square-unit, and whose vertex is one unit above the plane of the base.

a) 1/4
b) 1/3
c) 1/2
d) 2/3
e) 1

Solution:

Correct answer: b
One-third area of base times height – or by integral (areas parallel to base are proportional to square of distance from vertex.

25. Find the greatest common divisor of 123432123432100 and 2468642468642000015.

a) 1
b) 3
c) 5
d) 7
e) 15

Solution:

Correct answer: c
For $n = 123432123432100$ and $m = 2468642468642000015$, we have $n = 20000m + 15$ So a common divisor of n and m must divide 15. Both n and m is divisible by 5 since their last digit is 0 or 5. The number n is not divisible by 3 since the sum of the digits of n is not divisible by 3. Thus the greatest common divisor is 5.

26. Suppose that $f(x)$ is a polynomial such that $f(x+3) = x^2 + 4x$. What is the sum of the zeros of $f(x)$.

a) 5
b) 2
c) -2
d) 4
e) -3

Solution:

Correct answer: b
Since f is a polynomial, $f(x) = ax^2 + bx + c$. Hence $a(x+3)^2 + b(x+3) + c = x^2 + 4x$. By equating coefficients we can see that $a = 1$, $b = -2$, and $c = -3$. Hence $f(x) = x^2 - 2x - 3 = (x+1)(x-3)$. Hence $f(x)$ has roots 3 and -1.

27. A rectangle is partitioned into 4 subrectangles as shown below. If the subrectangles have the indicated areas, find the area of the unknown rectangle.

60	100
?	80

a) 72
b) 21
c) 70
d) 48
e) 64

Solution:

Correct answer: d

As in the figure to the right, $cb = 60$, $bd = 100$, and $ad = 80$.
Then $ac = ad \cdot \frac{c}{d} = 80 \cdot \frac{cb}{bd} = 80 \cdot \frac{60}{100} = 48$.

b	60	100
a	?	80
	c	d

28. How many pairs of integers (x, y) satisfy the following equations.

$$x^3 + 6x^2 + 8x = 3y^2 + 9y + 1.$$

a) 0 b) 1 c) 2

d) 3 e) None of the above.

Solution:

Correct answer: a
Notice that $x^3 + 6x^2 + 8x = x(x^2 + 6x + 8) = x(x+4)(x+2)$ which must be divisible by three because either $x, x+2, x+4$ is divisible by 3. However, $3y^2 + 9y + 1$ has a remainder of 1 when divided by 3. Thus there 0 pairs of integers satisfying the equation.

29. A bucket contains 8 red marbles and 4 blue marbles. If we select 3 marbles randomly from the bucket without replacement, what is the probability that no blue marbles are selected?

a) $\frac{7}{36}$ b) $\frac{14}{55}$ c) $\frac{7}{55}$

d) $\frac{14}{36}$ e) None of the above.

Solution:

Correct answer: b
There are ${}_{12}C_3 = 220$ ways of choosing 3 marbles and ${}_8C_3 = 56$ was of choosing 3 red marbles. Thus the probability is 56/220= 14/55.

30. Two 8.5-inch by 11-inch sheets of paper are laying on a 3 foot by 3 foot table. 1173 square inches of the table is not covered by paper. What is the area of the overlap between the two sheets of paper?

a) 64 sq inches b) 36 sq inches c) 187 sq inches

d) 32 sq inches e) 123 sq inches

Solution:

Correct answer: a
The table is 1296 square inches. Thus 123 square inches of the table is covered by paper. Since the area of two sheets of paper is 187 square inches. Thus the overlap must be 64 square inches.

31. A bag contains 12 coins consisting of quarters, dimes, and nickels. If the total value of the coins in the bag is at most \$2.15. What is the maximal number of quarters the bag can contain?

a) 7 b) 8 c) 5

d) 6 e) 9

Solution:

Correct answer: a
If the bag had 9 quarters then it would have at least \$2.25. If the bag 8 quarters, then it would have at least \$2.20. Thus the bag must have at most 7 quarters. If the bag has 7 quarters, 2 dimes, and 2 nickels then its value is \$2.05 which is less than \$2.15.

32. A group of airplanes is based on a small island. The tank of each plane holds just enough fuel to take it halfway around the world. Any desired amount of fuel can be transferred from the tank of one plane to the tank of another while the planes are in flight. The only source of fuel is on the island. It is assumed that there is no time lost in refueling either in the air or on the ground. The planes have the same constant speed and rate of fuel consumption. What is the smallest number of planes that will ensure the flight of one plane around the world on a great circle and have all the planes return safely to their island base?

a) 3
b) 4
c) 6
d) 7
e) 9

Solution:

Correct answer: a
Planes A, B, and C take off together. After going $\frac{1}{8}$ of the distance, C transfers $\frac{1}{4}$ tank to A and $\frac{1}{4}$ to B and C returns. Planes A and B continue another $\frac{1}{8}$ of the way, then B transfers $\frac{1}{4}$ of a tank to A and B returns. A flies $\frac{3}{4}$ the way around and is met by B who transfers $\frac{1}{4}$ of a tank to A. B turns around and follows A to the base. At $\frac{1}{8}$ the way from the base they are met by C who transfers $\frac{1}{4}$ of a tank to each and they all return to the base safely.

33. Amy likes either red or green clothes, but not both. She likes either turtleneck or V-neck sweaters, but not both. Amy never wears a sweater that is both the color and the type she likes, nor does she wear one that is neither the color nor the type she likes. Amy wears a red turtleneck. If you want to buy a sweater for Amy that she will wear. Should you buy a red V-neck, a green V-neck, or a green turtleneck?

a) A red V-neck
b) A green V-neck
c) A green turtle neck
d) Amy won't wear any of those sweaters
e) Not enough information is given to answer the question

Solution:

Correct answer: b
If Amy wears a red turtleneck, then she either likes red and V-neck sweaters or she likes green and turtleneck sweaters. In either case she will wear a green V-neck sweater.

34. Find the units digit of 13^{2017}.

a) 5
b) 7
c) 9
d) 1
e) 3

Solution:

Correct answer: e
The units digit of 13^{2017} is the same as that of 3^{2017}. Since $2017 = 4 \times 504 + 1$, we have

$$3^{2017} = 3^{4\times504+1} = 3^{4\times504}3 = (3^4)^{504}3 = 81^{405}3.$$

So the units digit is 3.

35. Find all real numbers a such that $f(x) = x^2 + a|x-1|$ is an increasing function on the interval $[0,\infty)$.

a) $[-2,\infty)$ b) $[-2,0]$ c) $(-\infty,0]$

d) $[0,2]$ e) $[-2,2]$

Solution:

Correct answer: b
For $0 \le x \le 1$, $f(x) = x^2 - ax + a$. Thus $f(x)$ is increasing on the interval $[0,1]$ if and only if $a \le 0$. On the interval $[1,\infty)$, $f(x) = x^2 + ax - a$. Hence, $f(x) = x^2 + ax - a$ is increasing on $[1,\infty)$ if and only if $a \ge -2$. Hence $[-2,0]$ is the set of real numbers such that the function is increasing.

36. Let $f(x)$ be an odd function defined on $\mathbb{R}$ satisfying

(a) $f(x+2) = -f(x)$, for all real numbers x;
(b) $f(x) = 2x$ when $0 \le x \le 1$.

Find the value of $f(10\sqrt{3})$.

a) $20\sqrt{3}-36$ b) $-20\sqrt{3}+36$ c) $-10\sqrt{3}+18$

d) $10\sqrt{3}+18$ e) $-20\sqrt{3}-36$

Solution:

Correct answer: b
Since $f(x+2) = -f(x)$, we have

$$f(x+4) = -f(x+2) = f(x).$$

Hence f is a periodic function with period 4. Then using that $f(x)$ is an odd function and $f(x) = 2x$ when $0 \le x \le 1$, we have $f(10\sqrt{3}) = f(10\sqrt{3}-16) = -f(-10\sqrt{3}+16)$

$$= f(-10\sqrt{3}+18) = 2(-10\sqrt{3}+18) = -20\sqrt{3}+36.$$

37. Assume x and y are real numbers and satisfy $x^3 + 2x^2y - 3y^3 = 0$. Then $x^2 + y^2$ must be equal to which of the following?

a) 2 b) $2x$ c) $2x^2$

d) $2x^3$ e) $2y$

Solution:

Correct answer: c

If $y \neq 0$, then dividing by y^3 we obtain $\frac{x^3}{y^3} + 4\frac{x^2}{y^2} - 3 = 0$. Letting $z = \frac{x}{y}$ this gives $z^3 + 2z^2 - 3 = 0$ which has a unique real solution $z = 1$. Thus $x = y$ and $x^2 + y^2 = 2x^2$.
If $y = 0$ then $x^3 = 0$. Thus $x = 0$ and $x^2 + y^2 = 2x = 0$.

38. $\sqrt[3]{2\sqrt{13}+5} - \sqrt[3]{2\sqrt{13}-5} =$

a) -1 b) $2\sqrt{13}$ c) 10

d) 1 e) 3

Solution:

Correct answer: d

Let $A = \sqrt[3]{2\sqrt{13}+5}$ and $B = \sqrt[3]{2\sqrt{13}-5}$. Then we wish to solve for $A - B$. Notice that $A^3 - B^3 = 2\sqrt{13} + 5 - (2\sqrt{13} - 5) = 10$ and $AB = \sqrt[3]{(2\sqrt{13}+5)(2\sqrt{13}-5)} = 3$.
$(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3 = A^3 - B^3 - 3AB(A-B) = 10 - 9(A-B)$
$(A-B)^3 + 9(A-B) - 10 = \big((A-B) - 1\big)\big((A-B)^2 + (A-B) + 10\big) = 0$. Since $A - B$ is real, $A - B = 1$.

39. Suppose real numbers x, y, z satisfy the equation $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 1$.

Compute the value of

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

a) 1 b) -1 c) 2

d) 0 e) -2

Solution:

Correct answer: d
We first note that $x + y + z \neq 0$. Otherwise,

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = -3.$$

Thus, we have

$$\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)(x+y+z) = (x+y+z).$$

Simplifying it, we have

$$\frac{x^2}{y+z} + x + \frac{y^2}{z+x} + y + \frac{z^2}{x+y} + z = (x+y+z),$$

which yields

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 0.$$

40. Find the area of overlap between the two circular discs, $x^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 3$.

a) $\frac{\pi}{2} - 1$

b) $\frac{\pi}{4}$

c) $\frac{\pi}{2} - \frac{\sqrt{3}}{2}$

d) $\frac{4}{5}$

e) $\frac{5\pi}{6} - \sqrt{3}$.

Solution:

Correct answer: e

The circles intersect in the points $(1/2, \pm\sqrt{3}/2)$. Draw a vertical line between those two points. Then the right half of the area of intersection is obtained by cutting out an isosceles triangle from the sector of 120 degrees (one third of a circle of radius 1). Thus its area is $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

The left half of the area of intersection is obtained by cutting out an isosceles triangle from the sector of 60 degrees (one sixth of a circle of radius $\sqrt{3}$). Thus its area is $\frac{\pi}{2} - \frac{3\sqrt{3}}{4}$. Add those areas together to get the correct area.

END OF EXAM