

Name: _____

State Math Contest - Senior Exam

Instructions:

- Do not turn this page until your proctor tells you.
 - Enter your name, grade, and school information following the instructions given by your proctor.
 - This is a multiple choice test with 40 questions. Each question is followed by answers marked a), b), c), d), and e). Only one answer is correct.
 - Mark your answer to each problem on the bubble sheet Answer Form with a #2 pencil. Erase errors and stray marks. Only answers properly marked on the bubble sheet will be graded.
 - SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
 - You will have 2 hours and 30 minutes to finish the test.
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1. If $4^x - 2^x - 6 = 0$, then $x =$

- a) $\log_2 3$ b) -2 c) $\log_4 3$ d) 3 e) $\log_3 2$

2. The sum of the infinite series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \dots$ equals

- a) 1 b) $\frac{9}{10}$ c) $\frac{4}{5}$ d) $\frac{7}{10}$ e) $\frac{3}{4}$

3. How many triplets (x, y, z) satisfy the following three equations?

$$\begin{aligned}3x + 2y + 4z &= 5, \\x - 2y - 3z &= 4, \\6x + 4y + 8z &= 10.\end{aligned}$$

- a) 1
b) 2
c) infinitely many
d) 0
e) a finite number, but more than 2.

4. There are 60 students in a class. Of the students, 36 are in club A, 26 students are in club B. The number of students that are in neither of the clubs is four more than one third of the number of students that are in both clubs. Find the number of students in both clubs.

- a) 5 b) 6 c) 7 d) 8 e) 9

5. The probability that it will rain at some point during 7:00-8:00 PM is 80%. The probability that it will rain at some point during 8:00 - 9:00 PM is also 80%. If these two probabilities are independent, what is the probability that it will rain at some point during 7:00 -9:00 PM?

- a) 64% b) 96% c) 80% d) 20% e) 16%

6. A polynomial of degree 4 whose coefficients are real numbers has zeros 1, 2, and $4 - i$. If the coefficient of x^4 is 1, what is the coefficient of x^3 ?

- a) -3 b) -8 c) -11 d) 19 e) 7

7. Find $\arctan 2 + \arctan 3$ in degrees.

- a) 120 b) 135 c) 150 d) 180 e) 315

8. A coin is flipped 8 times. What is the probability that the number of heads is strictly greater than the number of tails?

- a) $\frac{4}{9}$ b) $\frac{1}{2}$ c) $\frac{93}{256}$ d) $\frac{23}{84}$ e) $\frac{61}{84}$

9. A plane slices a cube so that the intersection is a polygonal region. What is the maximum number of sides of the polygonal region?

- a) 3 b) 4 c) 5 d) 6 e) 7

10. The four consecutive coordinates of a rhombus in the coordinate plane are (a, b) , (c, d) , (e, f) , and (g, h) , where $a \neq e$ and $c \neq g$. If $\log \frac{b-f}{a-e} = 1$, then $\log \frac{d-h}{g-c} =$

- a) -10 b) -1 c) $-\frac{1}{10}$ d) $\frac{1}{10}$ e) 10

11. I have letters to four people, and envelopes addressed to the four people, and (without looking) I randomly put a letter into each of the four envelopes. What is the probability that none of the letters will get put into its correct envelope?

- a) $\frac{1}{4}$ b) $\frac{3}{4}$ c) $\frac{81}{256}$ d) $\frac{3}{8}$ e) $\frac{23}{24}$

12. How many numbers in the range from 1 to 1,000, inclusive, are not divisible by the first three primes (2, 3 or 5)?

- a) 166 b) 266 c) 299 d) 701 e) 734

13. If a function f satisfies $f(n^2) = 2f(n) + 1$ and $f(2) = 3$, then $f(256)$ equals

- a) 6 b) 17 c) 24 d) 27 e) 31

14. If x is in the domain of the function $f(x) = \frac{(x^2 - 1)\sqrt{(3x - 2)}}{x - 1}$, then

- a) $0 < x < 2$ or $2 < x < \infty$
b) $-\infty < x < 1$ or $1 < x < \infty$
c) $\frac{2}{3} \leq x$
d) $\frac{2}{3} \leq x < 1$ or $1 < x < \infty$
e) x is any real number.

15. The expression $\frac{\sqrt{x+4} - \sqrt{x}}{4}$ is the same as

- a) $\frac{1}{\sqrt{x+4} + \sqrt{x}}$ b) $\frac{4}{\sqrt{x+4} + \sqrt{x}}$ c) $\frac{1}{2}$
d) $\frac{1}{\sqrt{x+4} - \sqrt{x}}$ e) $\frac{2x+4}{\sqrt{x+4} - \sqrt{x}}$

16. If we are to choose real numbers a and b such that $0 < a < b$, then we can best describe the solution set of $|x + 12| < |3x - 1|$ as the set of all x such that
- a) $-a < x < b$ b) $-b < x < a$
- c) $x < -a$ or $x > b$ d) $x < -b$ or $x > a$
- e) $x < a$ or $x > b$
17. Find the inverse function $f^{-1}(x)$ if $f(x) = \frac{2x - 1}{x + 3}$.
- a) $f^{-1}(x) = \frac{3x - 1}{2 - x}$ b) $f^{-1}(x) = \frac{3x + 1}{2 - x}$
- c) $f^{-1}(x) = \frac{2x + 1}{3 - x}$ d) $f^{-1}(x) = \frac{x + 1}{2 + 2x}$
- e) $f^{-1}(x) = \frac{2 - x}{1 + 3x}$
18. Train stations A and B are on the same railroad line and are 50 miles away from each other. A train leaves station A heading towards station B at 1:00 pm going 20 miles an hour. Another train leaves station B heading toward station A at 2:00 pm going 10 miles an hour. When will the two trains meet each other?
- a) 2:30 pm b) 3 pm c) 3:30 pm
- d) 4 pm e) 4:30 pm
19. If $x^a x^b = 1$ and $x > 1$, find $4a - b^2 + a^2 + 4b - 10$.
- a) -20 b) -10 c) 0 d) 10 e) 20
20. Find the number of digits in the product $25^{25} \times 2^{60}$.
- a) 76 b) 37 c) 54 d) 28 e) 65
21. Positive integers m and n satisfy the equation $(2m - 7)(2n - 7) = 25$. What are all possible values for $m + n$?
- a) 2, 20, 24 b) 3, 12, 16 c) 2, 10, 16
- d) 12, 20 e) 2, 12, 20
22. Three sides of a quadrilateral have lengths 3, 4, and 9. There exist positive real numbers a and b such that if l is the length of the fourth side of the quadrilateral, then $a < l < b$, and if l satisfies $a < l < b$, then there exists a quadrilateral with side lengths 3, 4, 9, and l . Find $a + b$.
- a) 18 b) 16 c) 9 d) 32 e) 24

23. Two square regions A and B each have area 8. One vertex of square B is the center point of square A . Find the area of $A \cup B$.
- a) 16 b) 15 c) $10\sqrt{2}$
- d) 14 e) Cannot be determined.
24. A plane slices a cone parallel to the base and one-third the distance from the vertex to the base and a second parallel plane slices the cone two-thirds the distance from the vertex to the base. What fraction of the volume of the cone is between the two slices?
- a) $\frac{8}{9}$ b) $\frac{7}{27}$ c) $\frac{1}{3}$ d) $\frac{1}{9}$ e) $\frac{2}{27}$
25. Suppose that the numbers 4-9, inclusive, are arranged in three pairs of distinct numbers. The numbers in each pair are added together, and the resulting three numbers are then multiplied together. What is the maximum value of the resulting product?
- a) less than 1,801 b) between 1,801 and 1,900, inclusive
- c) between 1,901 and 2,000, inclusive d) between 2,001 and 2,100, inclusive
- e) more than 2,100
26. Suppose that Miles lists all possible (distinct) rearrangements of the letters in the word MATHEMATICS. He then picks one rearrangement at random. What is the probability that the first five letters of this rearrangement are ATTIC (in order)?
- a) $\frac{1}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$ b) $\frac{2}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$
- c) $\frac{4}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$ d) $\frac{6}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$
- e) $\frac{8}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$
27. Suppose it costs \$ 1 to play the following game. You roll a die and flip a coin. If the coin lands on tails then you lose your dollar no matter what occurs with the die. If the coin lands on heads, and you roll an even number with the die, then you get your dollar back, but no more. If the coin lands on heads and you roll an odd number with the die, then you win your dollar back plus another \$5 dollars. On average, how much should you expect to win or lose (including the cost of playing) when playing this game?
- a) Win \$0.75 b) Win \$0.50 c) Break even d) Lose \$0.50 e) Lose \$0.75
28. Let S be the set of all positive integers n such that n^3 is a multiple of both 16 and 24. What is the largest integer that is a divisor of every integer n in S ?
- a) 6 b) 12 c) 18 d) 24 e) 216

29. Tyson is three times as old as Mandi. Two years ago, Tyson was four times as old as Mandi. How old is Mandi now?

- a) 16 b) 14 c) 7 d) 6 e) 4

30. Suppose that

$$A \star B = 2A + 3B, \text{ and } C \heartsuit D = \frac{C^2 + 2D}{D}.$$

What is $(12 \star 4) \heartsuit 3$?

- a) 432 b) 433 c) 434 d) 435 e) 436

31. Solution 1 contains only liquids a and b in a ratio of $1 : 4$. Solution 2 contains also contains only liquids a and b , but in a ratio of $1 : 1$. Solution 3 is obtained by mixing Solutions 1 and 2 in a ratio of $5 : 1$. How many Tablespoons of liquid a are in 60 Tablespoons of Solution 3?

- a) 15 b) 18 c) 20 d) 24 e) 30

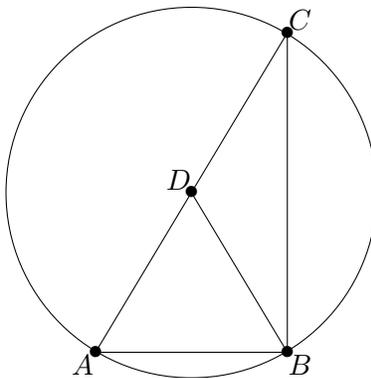
32. If the average test score for five students is 92, which of the following is the highest score a sixth student could get so that the average of all six scores would be no more than 86?

- a) 55 b) 56 c) 57 d) 58 e) 59

33. How many odd four-digit numbers are there that do not contain the digit 6?

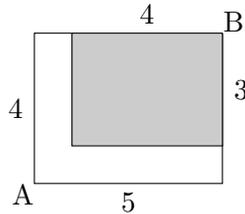
- a) 2560 b) 3240 c) 3645 d) 4050 e) 5000

34. Points A , B , and C lie on the circle. The point D is the center of a circle and lies on the line segment AC . If $AB = 6$ and $BD = 5$, find BC .

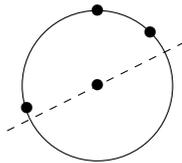


- a) 6 b) 6.5 c) 7 d) 7.5 e) 8

35. Two rectangles are given in the figure below so that two sides of the smaller rectangle are contained in two sides of the larger rectangle. The distances are given in miles. Anne and Bill both run from point A to point B each traveling at the same average speed without passing through the shaded smaller rectangle. Anne takes the lower route running straight to a corner of the shaded rectangle and then straight to B . Bill takes the upper route running straight to a corner of the shaded rectangle and then straight to B . If Anne takes 81 minutes, approximately how much longer will it take Bill to arrive?

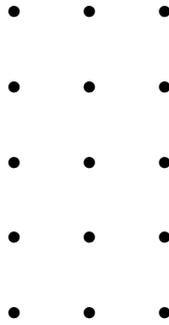


- a) 15 seconds b) 30 seconds c) 45 seconds
d) 1 minutes e) 4 minutes
36. Find the best estimate of the sum of the square roots of all the integers from 1 to 10,000, inclusive.
- a) 7,071 b) 57,740 c) 577,394 d) 666,667 e) 707,107
37. Let P be a third degree polynomial. If $P(1) = 1$, $P(2) = 1$, $P(3) = 7$, and $P(4) = 25$. Then $P(6) - P(5) =$
- a) 30 b) 42 c) 56 d) 60 e) 90
38. If three points are scattered randomly on a circle, what is the probability that one can draw a line through the center of the circle, such that all three points lie on one side of the line?



- a) $\frac{1}{2}$ b) $\frac{3}{4}$ c) $\frac{7}{8}$ d) $\frac{5}{8}$ e) $\frac{2}{3}$

39. How many isosceles triangles can be drawn if each vertex must be one of the dots in the following square lattice?



- a) at most 80 b) between 81 and 90, inclusive c) between 91 and 100, inclusive
- d) between 101 and 110, inclusive e) greater than 110
40. Annabeth and Elinor are hoping to meet for dinner. They will each arrive at their favorite restaurant at a random time between 6:00 and 8:00 pm, stay for 20 minutes, and leave. What is the probability that they will see each other at the restaurant?

- a) $\frac{7}{36}$ b) $\frac{9}{36}$ c) $\frac{11}{36}$ d) $\frac{13}{36}$ e) $\frac{15}{36}$