Utah State Mathematics Contest
Junior Exam
March 16, 2011

1. Making only vertical or horizontal moves between adjacent letters, in how many ways can you start from an *'M'* and move to the *'H'*, spelling the word *'MARCH'* ?

				M				
			M	A	**M**			
		M	A	R	A	**M**		
	M	A	R	C	R	A	**M**	
M	A	R	C	**H**	C	R	A	**M**
	M	A	R	C	R	A	**M**	
		M	A	R	A	**M**		
			M	A	**M**			
				M				

(a) 64 (b) 112 (c) 120 (d) 60 (e) 124

2. If the radius of a circle is increased by 200%, then the area will be increased by:

(a) 800% (b) 200% (c) 900% (d) 300% (e) 400%

3. Assuming $m \neq n, m \neq 0, n \neq 0$, $\frac{mn - m^2}{mn - n^2} - \frac{n^2 - m^2}{mn}$ reduced to lowest terms is equal to:

(a) 1 (b) $\frac{m}{n}$ (c) $\frac{2m^2 - n^2}{n}$ (d) $\frac{2m^2 - n^2}{m}$ (e) $-\frac{n}{m}$

4. Which of the following is a perfect square?

(a) $4! \cdot 9!$ (b) $9! \cdot 10!$ (c) $8! \cdot 10!$ (d) $8! \cdot 9!$ (e) $8! \cdot 9! \cdot 10!$

5. How many distinct lines make up the altitudes, medians and angle bisectors in a triangle that is isosceles but not equilateral?

(a) 3 (b) 7 (c) 9 (d) 5 (e) 6

6. If eight ducks eat eight bushels of oats in eight days, and if twelve geese eat twelve bushels of oats in twelve days, then approximately how many bushels of oats (to the nearest bushel) will twenty ducks and twenty geese eat in twenty days?

(a) 65 (b) 57 (c) 83 (d) 40 (e) 106

7. Each day, for three days, Randall withdraws exactly 20% of the money that was in his sock drawer at the start of the day. At the end of the third day, he had $30.72. How much money was in the sock drawer originally?

(a) $76.80 (b) $67.50 (c) $60.00 (d) $49.16 (e) $48.00

8. Dante wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

(a) 6 (b) 9 (c) 12 (d) 15 (e) 18

9. The equation $\frac{x+8}{5} = \frac{4x^2-48x}{x^2-12x}$ has how many distinct solutions?

(a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many

10. Assuming each of the following choices of successive discounts is available, which of the following choices is most advantageous to the consumer?

(a) two 15% discounts (b) five 6% discounts (c) three 10% discounts (d) six 5% discounts (e) one 30% discount

11. What is the largest number of acute angles that can be found in a convex decagon?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

12. A large group of puppies and kittens are waiting to be adopted. If 15 kittens are adopted, the ratio of puppies to kittens is now 3:1. Later, 24 puppies are adopted, putting the ratio of puppies to kittens at 1:3. How many animals were there originally?

(a) 51 (b) 117 (c) 99 (d) 85 (e) 78

13. Given a circle and two parallel lines both tangent to the circle, how many points are equidistant from all three objects? (assume all objects are contained in a plane)

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

14. How many of the following statements *must* be true for all real numbers a and b (where $a \neq b$ and $a \neq -b$)?

(i) $\frac{a}{(a-b)} < \frac{b}{(a-b)}$ (ii) $\frac{a}{(a+b)} < \frac{b}{(a+b)}$

(iii) $a(a-b) > b(a-b)$ (iv) $a(a+b) > b(a+b)$

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

15. A regular hexagon has one circle inscribed within and another circle circumscribed about it. What is the ratio of the area of the smaller circle to the area of the larger?

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{\sqrt{3}}{4}$ (e) $\frac{3}{4}$

16. For how many positive integers n does the sum $1 + 2 + \cdots + n$ evenly divide $6n$?

(a) 11 (b) 9 (c) 7 (d) 5 (e) 3

17. The top of a 25-foot ladder leans against the side of the school, with the base of the ladder 7 feet from the building. If the top of the ladder were to slide 4 feet down, the base of the ladder would slide:

(a) 8 ft (b) 10 ft (c) 5 ft (d) 7 ft (e) 4 ft

18. There is a positive integer, B, such that $2B$ is a perfect square, $3B$ is a perfect cube, and $5B$ is a perfect fifth (an integer to the fifth power). Assume that B is the smallest such number. How many factors are in the prime factorization of B?

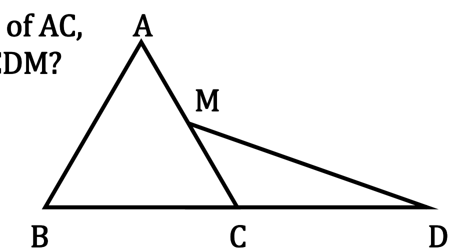
(a) 59 (b) 7 (c) 29 (d) 31 (e) 61

19. The number of terms in the expanded form of $[(x-2y)^2(x+2y)^2]^2$ when simplified is:

(a) 4 (b) 7 (c) 8 (d) 9 (e) 5

20. Equilateral ΔABC has side length 6. M is the midpoint of AC, and C is the midpoint of BD. What is the area of ΔCDM?

(a) $\frac{9\sqrt{2}}{2}$ (b) $\frac{27}{4}$ (c) $\frac{9\sqrt{3}}{2}$ (d) 8 (e) $9\sqrt{2}$



21. On a recent algebra exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. What is the difference between the mean and the median score on the exam?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

22. There are two values of p for which the equation $3x^2 + px - 5x + 8 = 0$ has exactly one distinct solution. What is the sum of those values?

(a) 3 (b) 7 (c) 4 (d) 6 (e) 10

23. The areas (in square inches) of the front, top, and side of a rectangular box are 3, 10, and 12, respectively. Which of the following choices is the closest to the measure of the volume of the box in cubic inches?

(a) 22 (b) 27 (c) 19 (d) 16 (e) 25

24. In which of the following cases are x and y neither directly nor inversely proportional?

(a) $\frac{2x}{5y} = 30$ (b) $2x - y = 8$ (c) $4xy = \sqrt{11}$ (d) $7x = 24y$ (e) $3x + 5y = 0$

25. The expression $\sqrt{5 + 2\sqrt{6}} - \sqrt{5 - 2\sqrt{6}}$ is equivalent to which of the following?

(a) $2\sqrt{6}$ (b) $2\sqrt{2}$ (c) $3\sqrt{5}$ (d) $3\sqrt{3}$ (e) $5\sqrt{2}$

26. Two trains are travelling on perpendicular tracks which pass through the same Union Station. The Pacific Railway is travelling north, while the Atlantic Zephyr is travelling east. As the Railway passes through Union Station, the Zephyr is 25 miles west of the station. After 20 minutes, the two trains are the same distance from the station. Another 80 minutes later, the two trains are again the same distance from the station. What is the ratio of the speed of the Zephyr to the speed of the Railway?

(a) $\frac{8}{5}$ (b) $\frac{4}{3}$ (c) $\frac{7}{5}$ (d) $\frac{3}{2}$ (e) $\frac{5}{3}$

27. The number of solution pairs that are positive integers to the linear equation $3x + 16y = 2011$ is:

(a) 38 (b) 39 (c) 40 (d) 41 (e) 42

28. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. Suppose that the integers x and y also satisfy the equation $x^2 - y^2 = z^2$ for some positive integer z. What is the value of $x + y + z$?

(a) 112 (b) 154 (c) 88 (d) 116 (e) 144

29. If the diagonals of a quadrilateral are perpendicular to one another, within which of the following classifications would the shape necessarily fall?

(a) a square (b) a rhombus (c) an isosceles trapezoid (d) a rectangle (e) none of these choices

30. Concrete sections on the new portions of I-15 are 80 feet long. As a car drives over the seams where sections meet, a brief thump is heard by the passengers. The speed of the car in miles per hour is approximately equal to the number of thumps heard in how many seconds?

(a) 40 (b) 85 (c) 45 (d) 55 (e) 70

31. Each student in a group of 80 students is right- or left-handed and right- or left-eyed (primary eye used for focusing). If 45 students are right-handed-right-eyed, 24 students are left-eyed, and 15 students are left-handed, then the number of left-handed-left-eyed students is:

(a) 3 (b) 4 (c) 0 (d) 6 (e) 2

32. Two numbers exist such that their difference, sum and product are to each other as 3:11:42. What is the larger of the two numbers?

(a) 16 (b) 6 (c) $\frac{8}{3}$ (d) 8 (e) $\frac{21}{2}$

33. For how many integer values of b can the expression $6x^2 + bx + 6$ be factored into *precisely* two prime binomial factors?

(a) 0 (b) 2 (c) 4 (d) 6 (e) 8

34. Millie buys a large number of hamburgers at a rate of 3 for $2 and the same number of corndogs at 5 for $4. If she sells all of the hamburgers and corndogs at a single rate, what price must she use to break even?

(a) 6 for $4.40 (b) 8 for $6.00 (c) 10 for $7.25

(d) 9 for $6.80 (e) 4 for $3.20

35. How many sets of two or more consecutive positive integers have a sum of 21?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

36. The graphs of the equations $x^2 + 2y = 16$ and $x + y = 8$ have two points of intersection. What is the distance between these two points?

(a) $2\sqrt{2}$ (b) $\sqrt{5}$ (c) $2\sqrt{3}$ (d) $2\sqrt{5}$ (e) 2

37. A barrel contains a selection of colored cubes, each of which is yellow, blue, or green. The number of green cubes is at least half of the number of blue cubes, and at most one third of the number of yellow cubes. The cubes which are blue or green number at least 34. The minimum number of yellow cubes is:

(a) 61 (b) 27 (c) 36 (d) 44 (e) 52

38. Imagine two circles, the larger with center M and radius m, the smaller with center N and radius n. Constructing line segment MN, which of the following choices cannot be true of the length of the line segment connecting the two centers?

(a) It is equal to $m - n$ (b) It is less than $m - n$ (c) It is equal to $m + n$

(d) It is less than $m + n$ (e) Any of these may be true

39. A parabola with the equation $y = x^2 + bx + c$ passes through the points $(1, 4)$ and $(5, 4)$. What is the value of c ?

(a) 9 (b) 0 (c) -2 (d) 5 (e) -4

40. Snowden left his entire estate to his four war buddies. He left half to Appleby and Huple, who shared their portion at a ratio of 5 to 4. Popinjay received twice as much as Huple. Lastly, Wintergreen received $4,000. How much did Popinjay receive?

(a) $18,000 (b) $20,000 (c) $16,000 (d) $32,000 (e) $24,000

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Junior Exam Solutions

1. (d) 60

Carving out one corner of the shape and re-orienting gives the following image. There are $2^4 = 16$ ways for the 'H' to trace out to the 'M'. Four times over would be 64, but that would be double-counting the far-left/right paths (the vertical/horizontal paths in the original image) each twice.

```
        H
      C   C
    R   R   R
  A   A   A   A
M   M   M   M   M
```

2. (a) 800%

Increasing the radius by 200% is equivalent to tripling the radius – not, as so many might instantly interpret, doubling it. Therefore, the area is nine times the original (or 800% bigger).

3. (e) $-\frac{n}{m}$

Using a common denominator of $mn(m-n)$, the expression is equivalent to $\frac{m^2n-m^3}{mn(m-n)} + \frac{m^3-m^2n-mn^2+n^3}{mn(m-n)} = \frac{-mn^2+n^3}{mn(m-n)} = \frac{(-1)(m-n)n^2}{mn(m-n)} = -\frac{n}{m}$.

4. (d) $8! \cdot 9!$

$8! \cdot 9! = 8! \cdot 9 \cdot 8! = (3 \cdot 8!)^2$

5. (b) 7

In a non-equilateral isosceles triangle, there is one unique angle. Because of the symmetry of an isosceles triangle, the angle bisector, median, and altitude coming from the unique angle are all the same line.

6. (c) 83

Each duck eats $\frac{1}{8}$ of a bushel every day. Each goose eats $\frac{1}{12}$ of a bushel every day. With twenty ducks and twenty days, there are 20·20=400 duck-days of feeding, and 20·20=400 goose-days of feeding. It follows that $400 \cdot \frac{1}{8} = 50$ bushels will be needed to feed the ducks, and $400 \cdot \frac{1}{12} \approx 33.3$ bushels will be needed to feed the geese.

7. (c) \$60.00

To withdraw 20% is equivalent to multiplying the original quantity by 0.80; to undo this three times would be equivalent to dividing by 0.80 three times. Dividing \$30.72 by $(0.80)^3$ yields an original amount of \$60.00.

8. (d) 15

There are three different ways to choose four identical donuts. There are three different ways to choose two matching pairs (like two glazed, two powdered) of donuts. There are six ways to choose one triplet and one single (like three glazed, one powdered). Lastly, there are three ways to choose one pair and two singles, for a total of fifteen combinations.

9. (a) 0

Factoring (and stipulating that $x \neq 12$ and $x \neq 0$) allows simplifying to $\frac{x+8}{5} = 4$. Solving yields $x = 12$, which is a contradiction. This equation has no solution.

10. (e) one 30% discount

Because subsequent discounts compound, each split reduces the effectiveness of the total discount. It's best to take the 30% as one large discount.

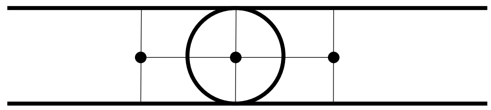
11. (b) 3

The sum of the exterior angles of any convex polygon is 360°. If any interior angle is acute, then the associated exterior angle must be obtuse, or greater than 90°. Since 4 or more obtuse angles would necessarily add to make more than 360°, there cannot be more than three acute interior angles. It is worth noting that the number of sides of the polygon is completely irrelevant to the problem.

12. (a) 51

Let P and K be the original numbers of puppies and kittens, respectively. Translating the given information yields $\frac{P}{K-15} = \frac{3}{1}$ and $\frac{P-24}{K-15} = \frac{1}{3}$. Solving these two equations gives $P = 27$ and $K = 24$, or 51 total animals.

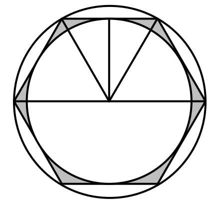
13. (c) 3



14. (b) 1

Options (i), (ii), and (iv) are easily contradicted. However, (iii) can be rewritten as $a^2 - ab > ab - b^2$, which is equivalent to $a^2 - 2ab + b^2 = (a-b)^2 > 0$, which is true for all real a, b where $a \neq b$.

15. (e) $\frac{3}{4}$



Splitting the hexagon up into 30-60-90 triangles, it is clear that the ratio of the radius of the inner circle to that of the outer is $\sqrt{3}$ to 2, making the ratio of areas 3 to 4.

16. (d) 5

The sum of the first n integers is equivalent to $\frac{n(n+1)}{2}$. For this to divide $6n$, $\frac{6n}{\frac{n(n+1)}{2}} = \frac{12}{(n+1)}$ must be an integer (where n is implicitly a positive integer). This is only true if n is contained within the set $\{1, 2, 3, 5, 11\}$.

17. (a) 8 ft

Using the Pythag. Thm., the top of the ladder currently rests 24 feet above the ground. If top of the ladder moves down from 24 feet to 20 feet, the base would have to shift from 7 feet to 15 feet away from the building (assuming the ladder remains intact).

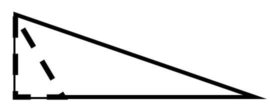
18. (a) 59

To minimize B, it must be assumed that the only possible prime factors are 2, 3 and 5. To meet the given conditions, $B = 2^{2m-1}3^{3n-1}5^{5p-1}$, where m, n, p are all positive integers. Also, $(2m-1)$must be a multiple of 3 and 5, $(3n-1)$ must be a multiple of 2 and 5, and $(5p-1)$must be a multiple of 2 and 3. For the smallest possible outcome, $(2m-1) = 15$, $(3n-1) = 20$, $(5p-1) = 24$, meaning $B = 2^{15}3^{20}5^{24}$, which has 59 total prime factors (not all distinct).

19. (e) 5

The expression can be rewritten as $\left(\left((x-2y)(x+2y)\right)^2\right)^2$. Under real number multiplication, the product of two first degree binomial conjugate factors results in two terms (after simplification). These two terms, squared and squared again, will give five resulting terms (after simplification).

20. (c) $\frac{9\sqrt{3}}{2}$



Drawing an altitude for △CDM, the small ghost shape is a 30-60-90 triangle. Given that the hypotenuse is 3, the height must be $\frac{3\sqrt{3}}{2}$. As well, CD must have length 6. Therefore, the area of △CDM is $\frac{1}{2} \cdot 6 \cdot \frac{3\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$.

21. (b) 1

35% of students got 70 or 80, and 55% of students got 70 or 80 or 85, meaning that the 50% mark falls in the category receiving 85 points, giving a median of 85. Adding up the sum of the scores times their respective proportions gives a mean of $0.1 \cdot 70 + 0.25 \cdot 80 + 0.20 \cdot 85 + 0.15 \cdot 90 + 0.30 \cdot 95 = 86$.

22. (e) 10

Quadratics have one distinct solution when the discriminant, $b^2 - 4ac = 0$. Therefore, $(p-5)^2 - 4 \cdot 3 \cdot 8 = 0$. Solving gives $p = 5 \pm \sqrt{96}$. Adding the two solutions for p gives $5 + \sqrt{96} + 5 - \sqrt{96} = 10$.

23. (c) 19

Let the length, width, and height be called L, W, H, respectively. From the given information, $L \cdot W = 10$, $L \cdot H = 3$, $W \cdot H = 12$. It follows that $L \cdot W \cdot L \cdot H \cdot W \cdot H = L^2W^2H^2 = (LWH)^2 = 360$. Thus the volume is $\sqrt{360} \approx 19$.

24. (b) $2x - y = 8$

Directly proportional variables share the relationship $y = kx$, while inversely proportional variables follow the form $xy = k$, where k is any real constant. All four other options can be solved to fit one of these two formats.

25. (b) $2\sqrt{2}$

Call $A = \sqrt{5 + 2\sqrt{6}}$ and $B = \sqrt{5 - 2\sqrt{6}}$. It follows that $A^2 = 5 + 2\sqrt{6}$, $B^2 = 5 - 2\sqrt{6}$, and $AB = \sqrt{25 - 24} = 1$. $(A - B)^2 = A^2 - 2AB + B^2 = 5 + 2\sqrt{6} - 2 + 5 - 2\sqrt{6} = 8$. Therefore, $A - B = \sqrt{8} = 2\sqrt{2}$.

26. (d) $\frac{3}{2}$

After 20 minutes, the trains are x miles west and north of the station, and after 80 more minutes, the trains are y miles east and north of the station, respectively. Given than the Pacific Railway started at the station, $y = x + 4x = 5x$. In the final 80 minutes, the PR travels a distance of $4x$, while the Atlantic Zephyr travels a distance of $6x$. In a fixed time, the ratio of speeds is equal to the ratio of distances, or $\frac{6x}{4x} = \frac{3}{2}$.

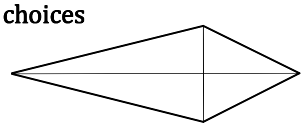
27. (e) 42

The solution with the smallest possible positive integer for y is $(665, 1)$. As the values must remain integers, further solutions can be achieved by subtracting 16 from x and adding 3 to y. The next solution found in this manner is $(649, 4)$. This method can continue until $(9, 124)$. This represents 41 additional solutions on top of the first.

28. (b) 154

Let $x = 10a + b$ and $y = 10b + a$, where a, b are one-digit positive integers. Then $x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a - b)(a + b)$. This can only be a square if 11 divides $(a - b)(a + b)$. Since these are one-digit numbers, $a + b = 11$. As well, $a - b$ is a square, and in this case must be 1. Now, $a = 6, b = 5$, leading to $x = 65, y = 56$, and $z = \sqrt{99 \cdot 11 \cdot 1} = 33$.

29. (e) none of these choices



30. (d) 55

$$\frac{x\ miles}{1\ hr} \cdot \frac{1hr}{3600\ sec} \cdot \frac{5280\ ft}{1\ mile} \cdot \frac{1\ thump}{80\ ft} = \frac{5280 \cdot x\ thumps}{80 \cdot 3600\ sec} = \frac{11x\ thumps}{600\ sec} \approx \frac{x\ thumps}{54.5\ sec}$$

31. (b) 4

From the given information, $x + z = 24$ and $y + z = 11$. As well, $x + y + z + 45 = 80$. Solving the system of equations gives: $x = 20, y = 11, z = 4$.

	Right-Eyed	Left-Eyed
Right-Handed	45	x
Left-Handed	y	z

32. (e) $\frac{21}{2}$

Let x be the larger number, and let y be the smaller. Then $\frac{x-y}{x+y} = \frac{3}{11}$ and $\frac{x+y}{xy} = \frac{11}{42}$. This yields $8x = 14y$ and $42x + 42y = 11xy$. Substituting gives $42x + 24x = \frac{44}{7}x^2$, or $21x = 2x^2$. Since x cannot be zero, it must be $\frac{21}{2}$.

33. (c) 4

For the trinomial to be factorable, there must be two integers which multiply to make 36 and add to b. However, it must also be the case that these two integers share no common factors, or else there would be a common factor amongst all terms in the expression. The only possible pairs are $(36, 1)$, $(9, 4)$, $(-9, -4)$, $(-36, -1)$. Therefore, b must be in the set $\{36, 13, -13, -37\}$.

34. (a) 6 for \$4.40

Hamburgers sell for $\$\frac{2}{3}$ each, and corndogs sell for $\$\frac{4}{5}$ each. Converting to a common denominator gives $\$\frac{10}{15}$ and $\$\frac{12}{15}$. The mean of these two rates is $\$\frac{11}{15}$, which is equivalent to 6 for \$4.40.

35. (c) 3

For the sum of n consecutive integers to equal X, it must be the case that: (1) if n is odd, n must divide X; (2) if n is even, n must divide $2X$. In this case, n can be 2, 3, 6, 7, 14, 21. However, if n is 7, 14, or 21, some of the integers will be negative. So, n can be 2, 3 or 6. The sets are $\{10, 11\}$, $\{6, 7, 8\}$, $\{1, 2, 3, 4, 5, 6\}$.

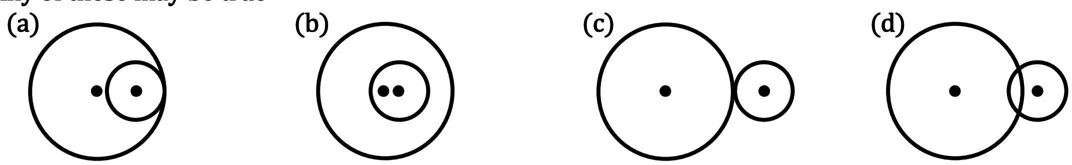
36. (a) $2\sqrt{2}$

Through substitution, $x^2 - 2x + 16 = 16$. Solving gives $x = 0$ or $x = 2$. The two points of intersection are $(0, 8)$ and $(2, 6)$. The distance between is $\sqrt{(0-2)^2 + (8-6)^2} = \sqrt{8} = 2\sqrt{2}$.

37. (c) 36

Let the number of yellow, green, and blue cubes be Y, G, B, respectively. Translating the given information (and multiplying out denominators) gives $2G \geq B$ and $3G \leq Y$. As well, $B + G \geq 34$. Substituting, $2G + G \geq B + G \geq 34$, or $3G \geq 34$. Since Y, G, B are positive integers, $G \geq 12$. Further, $Y \geq 3G \geq 36$.

38. (e) Any of these may be true



39. (a) 9

Substituting the given points, $b + c = 3$ and $5b + c = -21$, giving that $b = -6$ and $c = 9$.

40. (d) $32,000

Appleby and Huple have five and four shares, respectively. Given that Applyby's and Huple's shares total to be worth half of the inheritance, there are eighteen shares. Popinjay gets eight shares, and Wintergreen gets the final share worth $4000. Thus, Popinjay's eight shares are worth $32,000.