Utah State Mathematics Contest
Junior Exam
March 17, 2010

1. In the 2010 Vancouver Olympics, placement in curling semi-finals was based on a 10-team round-robin tournament, in which each of 10 teams plays each other team once, and the top four records move on to the semi-finals. A team is considered to be a strong contender for the gold medal if they win 7 of their 9 matchups. In this system, how many different teams can all end the round-robin tournament with a record of 7 or more wins?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

2. Determine the sum of the reciprocals of all positive integer factors of 28.

(a) 1 (b) $\frac{27}{14}$ (c) $\frac{3}{2}$ (d) $\frac{55}{28}$ (e) 2

3. Evan has the same birthday as his grandson, James. For the last six consecutive years, Evan's age was a multiple of James'. How old will Evan be at the next time that Evan's age will again be a multiple of his grandson's?

(a) 54 (b) 64 (c) 70 (d) 72 (e) 80

4. The repeating decimal of form of a number is $\mathbf{0.0\overline{48}}$... If such a number were converted to a fraction and reduced to lowest form, what would be the denominator?

(a) 165 (b) 110 (c) 55 (d) 990 (e) 22

5. What is the sum of all integers, M, for which $\mathbf{M^2 - 8M - 20}$ is a prime number?

(a) 2 (b) 4 (c) 6 (d) 8 (e) 10

6. Yarrrrrggghhhh! Literary critics complain that 80% of all fictional pirate captains have an eye-patch, 75% have a hook-hand, 67% have a peg-leg, and 90% have a pet parrot. By this accounting, at least what percent of these pirate captains must have an eye-patch, a hook-hand, a peg-leg *and* a parrot?

(a) 7% (b) 12% (c) 18% (d) 26% (e) 36%

7. Let $|\mathbf{U}-\mathbf{10}|=\mathbf{V},$ given that U < 10. What is the value of U – V?

(a) 10 – 2V (b) -10 (c) 10 – 2U (d) 10 (e) $|\mathbf{U}-\mathbf{10}|-\mathbf{U}$

8. Havermeyer recently inherited a sizeable sum of money. He paid 30% in taxes and invested 20% of what remained into M&M Enterprises. If M&M received $11,900 from Havermeyer, how much did Havermeyer pay in taxes?

(a) $18,200 (b) $19,600 (c) $17,850 (d) $25,500 (e) $3,570

9. The midpoints of the three sides of ΔABC are $(\mathbf{5},\mathbf{4})$, $(\mathbf{1},\mathbf{2})$, and $(-\mathbf{1},\mathbf{6})$. Which of the following is one of the actual points A, B, or C?

(a) $(\mathbf{4},\mathbf{9})$ (b) $(\mathbf{3},\mathbf{8})$ (c) $(\mathbf{5},\mathbf{2})$ (d) $(\mathbf{6},\mathbf{3})$ (e) $(\mathbf{0},\mathbf{6})$

10. Two tour guides are leading seven tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the condition that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

(a) 118 (b) 120 (c) 122 (d) 124 (e) 126

11. The positive integers C, D, C + D, C – D are all prime. The sum of these four primes must be divisible by which of the following?

(a) 2 (b) 3 (c) 5 (d) 7 (e) The sum is prime

12. Bill randomly selects three different numbers from the set {1, 2, 3, 4}, while Kerry randomly selects just one number from the set {2, 4, 6, 8, 10, 12}. What is the probability that Kerry's one number is larger than the sum of Bill's three numbers?

(a) $\frac{\mathbf{3}}{\mathbf{8}}$ (b) $\frac{\mathbf{1}}{\mathbf{3}}$ (c) $\frac{\mathbf{5}}{\mathbf{12}}$ (d) $\frac{\mathbf{5}}{\mathbf{8}}$ (e) $\frac{\mathbf{2}}{\mathbf{3}}$

13. A US military clock reads the time of day to be 2010 hours (this is 8:10 PM). At this time of day, what is the angle formed by the minute and hour hands of a 12-hour clock?

(a) 160° (b) 165° (c) 170° (d) 172° (e) 175°

14. For a fixed cube, how many unique planes contain at least three of the cube's vertices?

(a) 6 (b) 12 (c) 14 (d) 20 (e) 28

15. Which of the following equations, if graphed on a Cartesian plane, would have no y-intercept?

(a) $x^2 + y^2 = 2xy$ (b) $|y| + 4 = |x|$ (c) $\sqrt{x} = 6 - \sqrt{y}$

(d) $8^x + 8^y - 65 = 0$ (e) $\frac{x}{10} - \frac{10}{y} = \frac{(x+10)}{10}$

16. Alaskan oil reserves would potentially last for 35 years if used only by the United States. If the same reserves are also utilized by China, they will last for only 10 years. How many years would this oil last if only used by China?

(a) 15 (b) 21 (c) 25 (d) 14 (e) 20

17. An amalgam of mercury and cesium, 50 mL in volume, is found to be 30% cesium. A chemist wishes to combine this amalgam with 60 mL of pure cesium, but wants the final mixture to be exactly 60% cesium. To dilute the mixture, the chemist has a large supply of pure mercury. How many milliliters of pure mercury should be combined with the amalgam and pure cesium to produce the desired mixture?

(a) 0 (b) 5 (c) 10 (d) 15 (e) 20

18. A right triangle is inscribed in a circle with radius equal to $\frac{5}{\sqrt{2}}$. If the lengths of the legs of the triangle are distinct integers, what are these lengths?

(a) 8 & 4 (b) 9 & 3 (c) 7 & 1 (d) 6 & 5 (e) 11 & 3

19. The mean of three numbers is 7 more than the least of the numbers and 9 less than the greatest. The median of the three numbers is 6. What is their sum?

(a) 24 (b) 30 (c) 32 (d) 36 (e) 48

20. An arc of 60° on Circle A is half the length of an arc of 90° on Circle B. What is the ratio of the area of Circle A to the area of Circle B?

(a) $\frac{9}{16}$ (b) $\frac{3}{4}$ (c) $\frac{16}{9}$ (d) $\frac{9}{4}$ (e) $\frac{4}{3}$

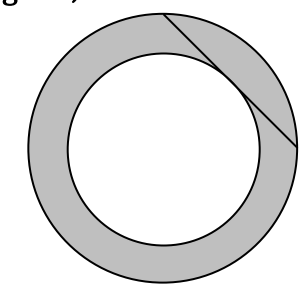
21. Compare the sum of all positive odd integers less than 2010 and the sum all positive even integers less than 2010. What is the difference between these two sums?

(a) 0 (b) 1005 (c) 2010 (d) 3015 (e) 4020

22. How many unique arrangements are there of the letters in the word UNUSUAL?

(a) 120 (b) 720 (c) 840 (d) 1260 (e) 1680

23. The two pictured circles have the same center, but different radii. The chord within the outer circle lies tangent to the inner circle. If the chord has length 6, what is the area of the shaded region?



(a) 3π (b) 12π (c) 6π (d) 18π (e) 9π

24. The number 761 satisfies the property that the digits fall in strictly decreasing order. How many three-digit numbers satisfy this property?

(a) 120 (b) 136 (c) 148 (d) 156 (e) 172

25. Which of the following triplets could be the side lengths of an obtuse scalene triangle?

(a) 8, 9, 12 (b) 7, 24, 25 (c) 19, 38, 57 (d) 15, 16, 17 (e) 10, 50, 51

26. If A and B are positive integers, where $A < B$, which fraction below is the largest?

(a) $\frac{A-1}{B-1}$ (b) $\frac{A^2-1}{B^2-1}$ (c) $\frac{A^3-1}{B^3-1}$ (d) $\frac{A+1}{B+1}$ (e) It depends on A & B

27. A line contains the points $P = (3, 5)$ and $Q = (19, 45)$. How many points on this line lie strictly between P and Q and have two integer coordinates?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

28. What is the area of the graph enclosed by $|5x| + |6y| = 30$?

(a) 7.5 (b) 15 (c) 30 (d) 36 (e) 60

29. An equilateral triangle has the same perimeter as a regular hexagon. What is the ratio of the area of the triangle to the area of the hexagon?

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{\sqrt{6}}{3}$ (d) $\frac{\sqrt{3}}{2}$ (e) $\frac{4}{9}$

30. Each problem on the 2010 Utah State Math Contest awards 5 points for a correct answer, 1 point for an omitted answer, and 0 points for a wrong answer. How many integer scores between 0 and 200 are unobtainable?

(a) 3 (b) 6 (c) 10 (d) 15 (e) 21

31. If $A = 3B$, $5B = 3C$, and $C = 2D$, then the ratio of $\frac{A}{4D}$ is equal to:

(a) $\frac{3}{4}$ (b) $\frac{8}{15}$ (c) $\frac{9}{10}$ (d) $\frac{5}{12}$ (e) $\frac{72}{5}$

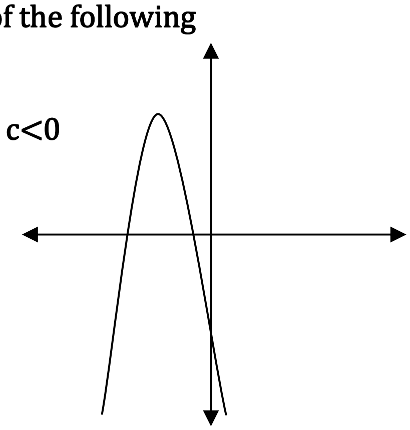
32. On a standard six-sided die, the six faces bear the numbers 1 to 6, and each number has an equal probability of being rolled. You roll the die until the sum of the numbers on your rolls exceeds 12. What is your most likely total?

(a) 13 (b) 14 (c) 15 (d) 16 (e) 13-18 are equally likely

33. If $2^{x+3} + 2^x = 3^{y+2} - 3^y$, and if x and y are integers, then $x + y$ is equal to:

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

34. If the graph of the equation $y = ax^2 + bx + c$ is shown, which of the following inequalities is false?



(a) $a < 0$ (b) $b < 0$ (c) $c<0$

(d) $\frac{-b}{2a} < 0$ (e) $b^2 - 4ac < 0$

35. Determine the constant term in the binomial expansion of the expression $\left(x^3 + \frac{3}{x^2}\right)^5$:

(a) 30 (b) 90 (c) 180 (d) 270 (e) 540

36. What is the 15th term in an arithmetic sequence in which the 3rd term is 20 and the 30th term is 2?

(a) 11 (b) 12 (c) 13 (d) 14 (e) 15

37. A pentagon has exterior angles 52°, (2x)°, 25°, (3x)°, and 38°. What is x?

(a) 13 (b) 31 (c) 49 (d) 67 (e) 85

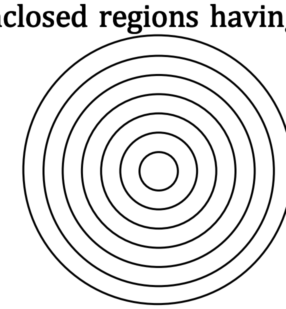
38. Suppose that the price of game tickets has gone up 4% every year for the past 25 years. If the price of a ticket this year is P, what was the price of a ticket 10 years ago?

(a) $\mathbf{0.6P}$ (b) $\frac{\mathbf{P}}{(\mathbf{1.04})^{\mathbf{10}}}$ (c) $\mathbf{P(0.6)^{10}}$ (d) $\frac{\mathbf{P}}{\mathbf{1.4}}$ (e) $\mathbf{P(0.96)^{10}}$

39. Copy editors look over documents for mistakes. Suppose that a document is checked over by two copy editors; the first editor finds 56 mistakes, and the second finds 50. Upon comparing their results, they find that only 40 mistakes have been found by both of them. For a given editor, each mistake has an equal likelihood of being noticed, though each editor has a different likelihood of catching any given mistake. Assuming that the two editors worked independently, what is the most likely count of the mistakes that both editors have missed?

(a) 4 (b) 9 (c) 12 (d) 14 (e) 20

40. The included picture contains seven concentric circles which have integer radii numbering from 1 to 7. How many non-congruent pairs of enclosed regions having equal area can be found in this picture?



(a) 0 (b) 1 (c) 2

(d) 3 (e) 4

Utah State Mathematics Contest
Junior Exam Solutions

1. (d) 5

First, each of the five top teams must each beat all of the five bottom teams. As well, each of the five top teams must beat exactly two of the other four top teams. If the teams are labeled A, B, C, D, E, then one such example would be:

A beats B, C B beats C, D C beats D, E D beats A, E E beats A, B

2. (e) 2

The number 28 is a perfect number, meaning that it is the sum of all of its proper factors. If the reciprocals are all converted to fractions with denominator 28, then the sum of the fractions would be $\frac{2 \cdot 28}{28} = 2$.

3. (c) 70

To meet the given conditions, Evan's age directly before the string of consecutive years must have been a multiple of 1, 2, 3, 4, 5, 6. The only human ages where this would be possible would be 60 and 120. Barring the extreme example (only two people in recorded history have been verified as reaching age 120), Evan is 60 years older than James. When Evan is 70, James will be 10.

4. (a) 165

Converting a decimal to a fraction depends on the number of digits until the repeating starts and the number of digits it takes to repeat. The given decimal is $\frac{48}{990} = \frac{8}{165}$.

5. (d) 8

$M^2 - 8M - 20 = (M - 10)(M + 2)$. These two factors have a difference of 12. This expression can only be prime if one of the factors is equal to 1 or -1 and if the other factor is a prime of the same sign. Thus, the two factors are either 1 & 13 or -1 & -13, leading to the conclusion that M is either 11 or -3.

6. (b) 12%

Let the probabilities of events A and B be P(A) and P(B), respectively. The minimum intersection of two given conditions is the lesser of (P(A) + P(B) – 1) and 0. The minimum intersection of N events is the lesser of *(the sum of the N probabilities minus (N – 1))* and 0. Thus, the minimum intersection of these four conditions is $(0.80 + 0.75 + 0.67 + 0.90) - 3 = 0.12$.

7. (a) 10 – 2V

If $U < 10$, then $|U - 10| = 10 - U = V$, so $U = 10 - V$, and $U - V = 10 - 2V$.

8. (d) $25,500

If the inheritance was K, the investment was (0.2)(0.7)K. Taxes were 0.3K, which is equal to $\frac{0.3}{0.14} \cdot 11,900 = 25,500$.

9. (b) (3,8)

If the three points are (A,D), (B,E), (C,F), then the three midpoints are $\left(\frac{A+B}{2}, \frac{D+E}{2}\right)$, $\left(\frac{B+C}{2}, \frac{E+F}{2}\right)$, $\left(\frac{A+C}{2}, \frac{D+F}{2}\right)$. This leads to the following systems of equations:

$$\begin{aligned} A + B &= 10 \qquad & D + E &= 8 \\ B + C &= 2 \qquad & E + F &= 4 \\ A + C &= -2 \qquad & D + F &= 12 \end{aligned}$$

Solving for the variables gives original points of (3,8), (7,0), (-5,4).

10. (e) 126

Each tourist has two choices, giving $2^7 = 128$ groups. However, excluding the two cases of all seven tourists choosing the same guide gives 126 groups.

11. (e) The sum is prime

(C – D), (C), (C+D) can only all be prime if D is even. Since D must also be prime, D is 2. As well, the only string of three consecutive odd numbers all being prime is 3, 5, 7 (every third odd must be a multiple of 3). So, the sum is 17.

12. (c) $\frac{5}{12}$

Bill's total must be 6, 7, 8 or 9. Kerry's probability of beating Bill's number is

$$\frac{0}{24}+\frac{0}{24}+\frac{0}{24}+\frac{2}{24}+\frac{4}{24}+\frac{4}{24}=\frac{10}{24}=\frac{5}{12}$$

13. (e) 175°

The hour hand rotates 30° every hour, or 0.5° every minute. At 8:10 PM, the hour hand would be 5° past 8. As there are 180° between 2 and 8, the hands would now be 5° closer.

14. (d) 20

The six faces of the cube each contain four vertices. The six planes connecting edge to opposite edge contain four vertices. Lastly, for each of the eight vertices, there is a plane containing all three of the adjacent vertices.

15. (b) $|y| + 4 = |x|$

If x is zero, the associated y-value is the y-coordinate of the y-intercept. In option (b), inserting a value of zero for x creates an equation with no real solutions.

16. (d) 14

Solving the problem involves combining rates. If the time to China to use the oil were T, then $\frac{1}{35}+\frac{1}{T}=\frac{1}{10}$. Solving the equation gives T = 14 years.

17. (d) 15

Let the amount of pure mercury required be X. The given conditions translate into $0.30 \cdot 50 + 1.00 \cdot 60 = 0.60 \cdot (50 + 60 + X)$. Solving gives that X must be 15mL.

18. (c) 7 & 1

If a right triangle is inscribed in a circle, then the hypotenuse of the triangle is the diameter of the circle, giving that the length of the hypotenuse is $\frac{5}{\sqrt{2}} \cdot 2 = 5\sqrt{2} = \sqrt{50}$. The only such right triangles with integer legs have legs of 7 & 1 or 5 & 5. As the legs are distinct, their lengths are 7 & 1.

19. (a) 24

Let M be the mean. $(M - 7) + (6) + (M + 9) = 3M$, giving M = 8. Thus, the sum is 24.

20. (a) $\frac{9}{16}$

Let P be the radius of Circle A, and R be the radius of Circle B. A's arc has length $\frac{\pi P}{3}$, and B's arc has length $\frac{\pi R}{2}$. Since $\frac{\pi P}{3} = \frac{1}{2} \cdot \frac{\pi R}{2}$, $P = \frac{3}{4}R$. The ratio of areas is $\frac{P^2}{R^2} = \frac{9}{16}$.

21. (b) 1005

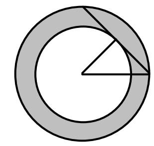
The sum of the first N odd integers is N^2. The sum of the first N even integers is N(N+1). The difference between the first 1005 odd integers and the first 1004 even integers is $1005^2 - 1004 \cdot 1005 = 1005$.

22. (c) 840

There are 3 U's, 1 N, 1 S, 1 A and 1 L. The number of arrangements is $\binom{7}{3,1,1,1,1} = \frac{7!}{3!1!1!1!1!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot1\cdot1\cdot1\cdot1} = \mathbf{840}.$

23. (e) 9π

Let the radii of the inner and outer circles be P and R, respectively. The shaded area is equal to $\pi R^2 - \pi P^2$. Because the chord is tangent to the inner circle, the pictured triangle is a right triangle, and $R^2 - P^2 = 3^2$. Thus, $\pi R^2 - \pi P^2 = 9\pi$.



24. (a) 120

There are $\binom{10}{3} = \mathbf{720}$ three-digit integers with three different digits. For any given three distinct digits, there six arrangements, only one of which would have the digits in decreasing order. So, there are 120 digits meeting the given criteria.

25. (e) 10, 50, 51

Any side of any triangle must be less than the total of the lengths of the other two sides. For a triangle to be obtuse, the sum of the squares of the two shortest sides must be less than the square of the largest. The only choice of side lengths meeting these requirements are 10, 50, 51.

26. (d) $\frac{A+1}{B+1}$

By factoring differences of squares or cubes, options (b) and (c) are both products of option (a) and a proper fraction, making options (b) and (c) both smaller than option (a). When the numerator and denominator of a fraction are of the same sign and are increased by the same positive constant, the result is closer to 1 than the original. As all of the fractions are proper fractions, option (d) is greater than option (a).

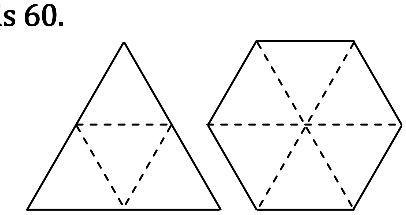
27. (c) 7

The difference in x-values is 16, and the difference in y-values is 40. The greatest common factor of 16 and 40 is 8. Thus, the line segment $\overline{PQ}$ can be separated into eight equal intervals having integer endpoints. These 8 intervals are defined by P, Q, and seven points in between.

28. (e) 60

The graph forms a rhombus with vertices at (6,0), (0,5), (-6,0), (0,-5). This rhombus can be divided into four congruent right triangles, each of which has height 5 and width 6. The area of each triangle is 15, so the total area is 60.

29. (a) $\frac{2}{3}$



The pictures shapes have the same perimeter, and are each composed of congruent miniature triangles. The ratio of areas is 4 to 6, or 2 to 3.

30. (b) 6

With more than 36 right answers, there are not enough remaining problems to leave blank to attain some of the scores in between multiples of 5. Scores of 199, 198, 197, 194, 193, and 189 are not attainable.

31. (c) $\frac{9}{10}$

$\frac{A}{3B} \cdot \frac{5B}{3C} \cdot \frac{C}{2D} = \frac{5A}{18D} = \mathbf{1}. \frac{A}{4D} \div \frac{5A}{18D} = \frac{18}{20} = \frac{9}{10}$.

32. (a) 13

The game ends when the total of all rolls exceeds 12. Any roll that might reach a specific number above 13 has an equally likely chance of reaching exactly13; therefore, 13 is at least as likely an outcome as any number above 13. On the other hand, a player advancing from a total of 7 has a possibility of reaching 13 (but nothing above 13) on their next roll. Therefore, the probability of ending with a total of 13 is higher (even if only slightly) than ending with any higher number.

33. (a) 5

Factoring the equation gives $2^x(2^3+1) = 3^y(3^2-1)$, or $2^x(9) = 3^y(8)$. It is only possible for x & y to be integers if they are 3 and 2, respectively. Therefore, $x + y = 5$.

34. (e) $b^2 - 4ac < 0$

The direction of the parabola makes $a < 0$. The y-intercept makes $c < 0$. The x-coordinate of the vertex makes $\frac{-b}{2a} < 0$, which also forces $b < 0$. The discriminant, on the other hand, must be positive, as this graph has two x-intercepts.

35. (d) 270

The 3rd term of the binomial expansion is $\binom{5}{2}(x^3)^2\left(\frac{3}{x^2}\right)^3 = (10)(x^6)\left(\frac{27}{x^6}\right) = 270$.

36. (b) 12

The difference between terms is $\frac{2-20}{30-3} = -\frac{18}{27} = -\frac{2}{3}$. Solving $\frac{X-20}{15-3} = -\frac{2}{3}$ gives $X = 12$.

37. (c) 49

The sum of the exterior angles of any polygon is 360°. Solving $(52)°+(2x)°+(25)°+(3x)°+(38)°= 360°$ yields $x = 49$.

38. (b) $\frac{P}{(1.04)^{10}}$

Increasing P by 4% is the same as multiplying P by 1.04. Taking this process back in time 10 years would be equivalent to dividing by 1.04 exactly 10 times.

39. (a) 4

The number of mistakes caught by at least one editor is 56 + 50 – 40 = 66. Let X be the number mistakes missed by both editors, meaning that the total number of mistakes in the document is 66+X. The probability that the first editor catches a particular mistake is $\frac{56}{66+X}$, while the probability that the second editor catches a particular mistake is $\frac{50}{66+X}$. Because the two editors' catches are independent, $\frac{56}{66+X} \cdot \frac{50}{66+X} = \frac{40}{66+X}$. Solving this equation gives $X = \frac{56 \cdot 50}{40} - 66 = 70 - 66 = 4$.

40. (d) 3