Utah State Mathematics Contest
Senior Exam
March 18, 2009

1. A metal cylinder is melted down and reshaped into a new cylinder. If the new diameter has been decreased by 20% without changing the volume, then the height has been increased by what percent?

   (a) 20 (b) 25 (c) 36 (d) 46.75 (e) 56.25

2. John has coins that may include pennies, nickels, dimes, or quarters. The mean value of the coins is 20 cents. If he were to add one quarter to his money, the new mean value would be 21 cents. How many quarters did he originally have?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

3. If the quadratic equation $x^2 - px - q = 0$ has two distinct real roots, then:

   (a) $p^2 - 4q \geq 0$ (b) $-p^2 - 4q > 0$ (c) $p^2 + 4q \geq 0$
   (d) $p^2 > 4q$ (e) $p^2 > -4q$

4. Evaluate the following expression:

$$\log_2 9 \cdot \log_3 25 \cdot \log_5 4$$

   (a) 2 (b) 6 (c) 8 (d) 9 (e) 30

5. How many positive integers less than 100,000 are both perfect squares and perfect cubes?

   (a) 6 (b) 7 (c) 8 (d) 9 (e) 10

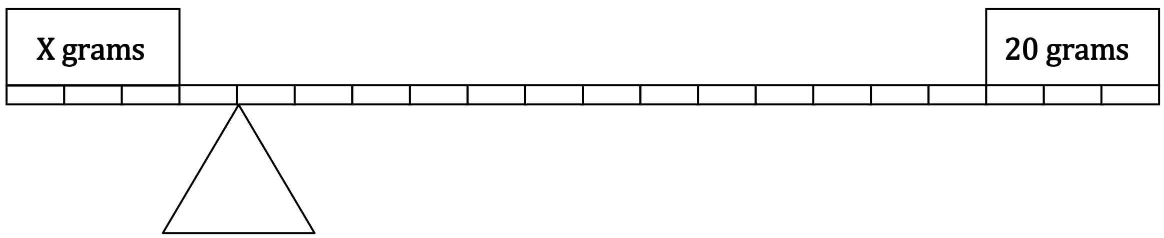
6. In the November 2008 general election, Utah voters turned out in strong numbers. Between Utah and Salt Lake counties, 67% of the 840 thousand registered voters made it to the election booth. Of registered voters, 65% of those in Salt Lake County voted, while 72% of those in Utah County voted. How many thousands of registered voters were in Utah County?

   (a) 230 (b) 240 (c) 250 (d) 260 (e) 270

7. When a line is drawn through a blank plane, it divides the plane into two sections. Further lines may divide the plane into more and more sections. Find the difference between the largest and smallest number of sections that may be created by drawing 9 distinct lines through a plane.

(a) 8 (b) 27 (c) 28 (d) 35 (e) 36

8. For what value of X would this system balance? (assume the lever has negligible mass)



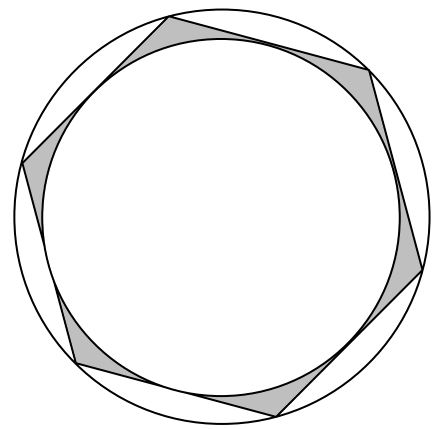
a) 148 (b) 260 (c) 116 (d) 140 (e) 80

9. How many distinct real solutions does the following equation have?

$$(4x^2 - 15x + 10)^{(4x^2 + 5x + 1)} = 1$$

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

10. The picture to the right illustrates a circle inscribed in a regular hexagon inscribed in another circle. What is the ratio of the combined areas of the shaded regions to the combined areas of the unshaded regions?



(a) $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$ (b) $\frac{4\pi - 6\sqrt{3}}{12\sqrt{3} - \pi}$ (c) $\frac{6\sqrt{3} - 3\pi}{6\pi - 4\sqrt{3}}$

(d) $\frac{6\sqrt{3} - 3\pi}{4\pi - 6\sqrt{3}}$ (e) $\frac{1}{12}$

11. If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then A^4 is equal to:

(a) $\begin{bmatrix} 16 & 32 \\ 0 & 16 \end{bmatrix}$ (b) $\begin{bmatrix} 16 & 16 \\ 0 & 16 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 4 \\ 0 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 8 & 16 \\ 0 & 8 \end{bmatrix}$ (e) $\begin{bmatrix} 16 & 1 \\ 0 & 16 \end{bmatrix}$

12. In the binomial expansion of $(x+y)^9$, what is the coefficient of the x^4y^5 term?

(a) 96 (b) 120 (c) 180 (d) 126 (e) 90

13. When written out in decimal form, how many zero-digits appear at the end of (2009!)?

(a) 200 (b) 222 (c) 401 (d) 500 (e) 2001

14. When Greg swims out from the beach, he is carried by the tide; it takes him 4 minutes to reach the nearest buoy. When he swims back in, he takes 16 minutes to swim against the tide. If Greg swims 100 yards per minute in still water, how many yards away is the buoy?

(a) 400 (b) 480 (c) 640 (d) 720 (e) 800

15. If $a > 1$, $\sin(x) > 0$, and $\log_a(\sin x) = c$, then $\log_a(\csc^2 x)$ is equal to:

(a) $\sqrt{1 - c^2}$ (b) $\frac{1}{c^2}$ (c) $-2c$ (d) c^{2a} (e) none of these

16. Find B if:

$$B = 1 + \cfrac{1 + \cfrac{1 + \cfrac{1+\cdots}{2+\cdots}}{2 + \cfrac{1+\cdots}{2+\cdots}}}{2 + \cfrac{1 + \cfrac{1+\cdots}{2+\cdots}}{2 + \cfrac{1+\cdots}{2+\cdots}}}$$

(a) $\frac{8}{5}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\frac{1 \pm \sqrt{5}}{2}$ (d) $\frac{\sqrt{5}-1}{2}$ (e) ∞

17. What is the remainder when $(13x^{10} - 6x^7 + 12x^3 - 3x - 5)$ is divided by $(x + 1)$?

(a) 3 (b) 5 (c) 7 (d) 9 (e) 11

18. Let $A = \begin{bmatrix} x & 5 & 12 \\ 0 & 1 & x \\ 2 & 0 & x \end{bmatrix}$. What is the sum of all of the values of x for which A^{-1} doesn't exist?

(a) -4 (b) -6 (c) -8 (d) -9 (e) -10

19. Given that $7^y = 49^{x+6}$ and $32^x = 16^{y-9}$, then difference between x and y is:

(a) 8 (b) 9 (c) 12 (d) 15 (e) 16

20. Determine which of the following complex numbers is a solution to the equation:

$$x^6 + 1 = 0$$

(a) $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ (b) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (c) $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ (d) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ (e) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

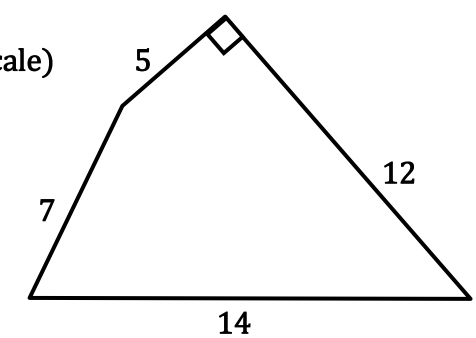
21. For some base, G, the number 359 would appear as 2009 in base 10. If $M = \frac{G-1}{3}$, what is the base M representation of 2009?

(a) 4027 (b) 2672 (c) 2009 (d) 3731 (e) 5200

22. Suppose that six brothers (Kojo, Largo, Milo, Nico, Otto, and Paulo) each roll a fair 6-sided die. At least three of them have rolled an even number. What is the probability that Largo, Paulo, and Milo have each rolled an even number?

(a) $\frac{7}{22}$ (b) $\frac{3}{32}$ (c) $\frac{1}{2}$ (d) $\frac{4}{17}$ (e) $\frac{4}{21}$

23. The quadrilateral pictured (not necessarily drawn to scale) contains one right angle, two acute angles, and one obtuse angle. Determine the area, in square units, of the given shape.



(a) $14\sqrt{30}$ (b) $30 + 2\sqrt{510}$ (c) $30 + \frac{1}{2}\sqrt{8281}$

(d) $30 + 14\sqrt{13}$ (e) $30 + 6\sqrt{38}$

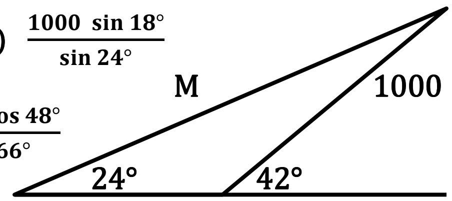
24. At 12:00 PM, the second, minute and hour hand on a clock all point to XII. Assuming continuous motion of all three hands, how many times will the second hand pass the minute hand in the time it takes for the minute hand to pass the hour hand three times?

(a) 192 (b) 193 (c) 194 (d) 195 (e) 196

25. What is the length of side M in the given triangle (not drawn to scale)?

(a) $1000 \csc 24°$ (b) $\frac{1000}{\cos 18°}$ (c) $\frac{1000 \sin 18°}{\sin 24°}$

(d) $\frac{1000 \sin 42°}{\cos 24°}$ (e) $\frac{1000 \cos 48°}{\cos 66°}$



26. In an arithmetic sequence adding to 2009, the sum of the two least elements is one third of the sum of the three greatest. If the sequence has seven elements, determine the largest.

(a) 287 (b) $669\frac{2}{3}$ (c) 410 (d) 533 (e) $502\frac{1}{4}$

27. The following table includes the scoring totals for a six-match soccer tournament amongst four MLS teams (each team plays each other team once). In each match, a team gets two tournament points for a win and 0 points for a loss. If there is a tie, each team gets one tournament point.

TEAMS	Goals For	Goals Against	Tournament Points
DC United	2	1	4
LA Galaxy	2	5	1
New England Revolution	4	0	5
Real Salt Lake	1	3	2

NE vs DC	___ to ___	DC vs RSL	___ to ___
NE vs RSL	___ to ___	DC vs LA	___ to ___
NE vs LA	___ to ___	RSL vs LA	___ to ___

Determine the score in the match between DC United and Real Salt Lake (<u>DC to RSL</u>).

(a) 0 to 0 (b) 1 to 0 (c) 2 to 0 (d) 1 to 1 (e) 2 to 1

28. Mara and Lara are in a shooting competition. The object of the match is to be the first to hit the bullseye of a target at 100 feet. Each opponent has a 40% chance of hitting the bullseye on a given shot. If Lara graciously allows Mara to shoot first, what is the probability that Lara will win the competition?

(a) $\frac{2}{5}$ (b) $\frac{3}{10}$ (c) $\frac{3}{8}$ (d) $\frac{7}{20}$ (e) $\frac{13}{36}$

29. Find the sum of the following infinite series:

$$\frac{1}{4} + \frac{2}{16} + \frac{3}{64} + \frac{4}{256} + \ldots$$

(a) $\frac{4}{7}$ (b) $\frac{3}{5}$ (c) $\frac{7}{15}$ (d) $\frac{1}{2}$ (e) $\frac{4}{9}$

30. What is the sum of the zeros of the function: $f(x) = x^4 - 6x^3 + 13x^2 - 14x + 6$?

(a) 0 (b) 4 (c) 6 (d) 9 (e) 14

31. There is a work crew that consists of 20% electricians, 30% plumbers, and 50% engineers. Amongst these people, 60% of the electricians have college degrees, 40% of the plumbers have degrees, and 80% of the engineers have degrees. If a randomly selected member of the crew has a college degree, what is the probability that this person is a plumber?

(a) $\frac{3}{25}$ (b) $\frac{3}{11}$ (c) $\frac{2}{9}$ (d) $\frac{3}{16}$ (e) $\frac{3}{10}$

32. Assume that you have 12 poles, each measuring a different whole number of meters ranging from 1 meter to 12 meters in length. How many distinct combinations of 3 poles can join end to end to form a triangle?

(a) 72 (b) 95 (c) 87 (d) 154 (e) 100

33. If x and y are real numbers such that $\frac{2y}{9x}$, $\frac{x}{2y}$, and $(12x - 8y)$ are all equivalent quantities, then $(9x + 6y)$ is equal to:

(a) -6 (b) -4 (c) $-\frac{9}{4}$ (d) $-\frac{3}{2}$ (e) 0

34. A researcher took a random sampling of 20 students and asked the amount of cash they had in their wallets. He found that the contents of the students' wallets had a mean of \$42, as well as a median of \$42. There were no perceived outliers in the group.
At this point, the researcher realized he had left a 21st student off of his calculations. The 21st student had \$80. After recalculating statistics for all 21 students, which of the following statements must be true?

(I) The mean increased
(II) The median increased
(III) The range increased
(IV) The mean is greater than the median

(a) I (b) I & IV (c) I, II & IV (d) I, III & IV (e) I, II, III & IV

35. In choosing a random real number, x, between 0 and 10, what is the probability that the following inequality holds true? (x is measured in radians)

$$\cot x \geq 1$$

(a) $\frac{40 - 9\pi}{40}$ (b) $\frac{\pi - 1}{10}$ (c) $\frac{4 - \pi}{4}$ (d) $\frac{40 - 6\pi}{40}$ (e) $\frac{3\pi}{40}$

36. Evaluate the limit: $\lim_{b \to \infty} \int_1^b \frac{x}{x^4 + 1}\, dx$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$ (e) $\frac{\pi}{10}$

37. There exists some positive real number, M, such that if you add five M's together, you get the same result as if you were to multiply five M's together. Which of the following is true about M?

(I) M^{15} is rational (II) M^{12} is rational (III) M^{30} is rational

(a) I only (b) II only (c) III only (d) II & III (e) I & III

38. What is the maximum value of B that would allow a real solution to the following equation? $2x^2 + 3y^2 - 18x - 18y + B = 0$

(a) 162 (b) $\frac{135}{4}$ (c) $\frac{135}{2}$ (d) $\frac{117}{4}$ (e) $\frac{117}{2}$

39. At the point specified by $\theta = \frac{\pi}{6}$, determine the slope of the line tangent to the curve:

$$r = 2 - \sin\theta$$

(a) $\frac{4-3\sqrt{3}}{11}$ (b) $\frac{-\sqrt{3}}{3}$ (c) $\frac{1-\sqrt{3}}{6}$ (d) $\frac{1-2\sqrt{3}}{3}$ (e) $\frac{-\sqrt{6}}{2}$

40. Five women are playing a card game in which they each have five cards. Each card is one of five different colors. All of the following statements are true:

(I) Any player who has a yellow card also has an orange card.
(II) Only if a player has a yellow and a red card does she have a blue card.
(III) A player has a green card if she does not have a yellow card.
(IV) A player does not have a blue card only if she does not have an orange card.
(V) Of blue, green, yellow and orange cards, each player has at least two colors.
Only one player currently has one card of each color. This woman is the only player holding a card that is:

(a) Red (b) Orange (c) Yellow (d) Green (e) Blue

2009 Utah State Mathematics Contest
Senior Exam Solutions

1. (e) 56.25

   $V = \pi r^2 h$. If the diameter is down 20%, the radius is down 20%. To maintain the same volume, the equation describing the adjusted lengths would be $V = \pi(\frac{4}{5} r)^2 (\frac{25}{16} h)$. Since $\frac{25}{16} = 1.5625$, there must be an increase of 56.25%.

2. (d) 3

   Let X be the initial number of coins. Since their average value is 20 cents, their total value is 20X. By adding a quarter, we get $\frac{20X + 25}{X + 1} = 21$. Solving for X, there were originally 4 coins, totaling 80 cents in value. They must have been three quarters and a nickel.

3. (e) $p^2 > -4q$

   The discriminant of the quadratic formula is $(b^2 - 4ac)$, which in this case is $(p^2 + 4q)$. For a quadratic equation to have two distinct real roots, $p^2 + 4q > 0$, or $p^2 > -4q$.

4. (c) 8

   $\log_2 9 \cdot \log_3 25 \cdot \log_5 4 = \frac{\log 9}{\log 2} \cdot \frac{\log 25}{\log 3} \cdot \frac{\log 4}{\log 5} = \frac{2\log 3}{\log 2} \cdot \frac{2\log 25}{\log 3} \cdot \frac{2\log 2}{\log 5} = 2 \cdot 2 \cdot 2 = 8$

5. (a) 6

   For an integer to be both a square and a cube, it must be a perfect sixth power. Positive integers to the sixth power include $1^6 = 1$; $2^6 = 64$; $3^6 = 729$; $4^6 = 4096$; $5^6 = 15625$; $6^6 = 46656$; $7^6 = 117649$... only the first six positive integers to the sixth power are less than 100,000.

6. (b) 240

   Let X be the number of voters from Salt Lake County, and Y be the number of voters from Utah County, both measured in thousands. $X + Y = 840$, and $0.65X + 0.72Y = 0.67(840)$. Solving this system of equations yields that $X = 600$, $Y = 240$.

7. (e) 36

   The minimum number of sections is 10 (draw 9 parallel lines). To achieve maximum number of sections, each successive line must cross all existing lines. The first line creates 2 sections; the second line cross the first and leads to 4 sections; the third line crosses the first 2 and leads 7 sections. The N^{th} new line can create N new sections. Beginning with 1 section (the plane), and adding $1 + 2 + ... + 8 + 9$ new sections gives 46 maximum sections. $46 - 10 = 36$.

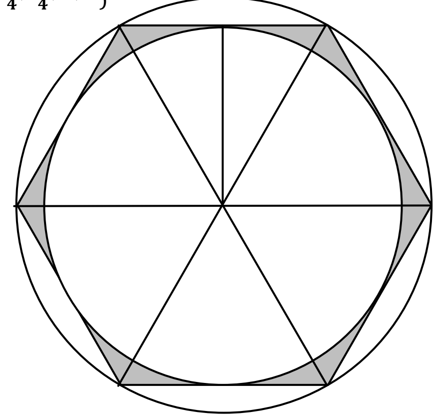
8. (c) 116

   The center of mass of the left weight is 2.5 steps away from the fulcrum, while the center of mass of the right weight is 14.5 steps away. It must be the case that $X \cdot 2.5 = 20 \cdot 14.5$. Solving for X gives 116.

9. (d) 5

   There are three possible situations: (I) the exponent polynomial is equal to zero, (II) the base polynomial is equal to one, or (III) the base polynomial is equal to (-1) and the exponent is an even integer. For (I), the two solutions are (-1) and $\left(-\frac{1}{4}\right)$. For (II), the two solutions are 3 and $\frac{3}{4}$. For (III), both 1 and $\frac{11}{4}$ would set the base exponential to (-1), but $\frac{11}{4}$ would make the exponent an odd integer. So, the 5 solutions are: $\left\{-1, -\frac{1}{4}, \frac{3}{4}, 1, 3\right\}$.

10. (a) $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$

    A regular hexagon can be divided into six equilateral triangles. Each side length is equal to the larger radius, R. The height of each equilateral triangle is the smaller radius, equal to $\frac{\sqrt{3}}{2}R$. The shaded area is equal to the difference between the area of the hexagon and the area of the smaller circle $= 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} R^2 - \frac{3\pi}{4} R^2$. The unshaded area is equal to the area of the large circle minus the shaded area $= \pi R^2 - (6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} R^2 - \frac{3\pi}{4} R^2)$. The ratio of the two areas is, when simplified, equal to $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$

11. (a) $\begin{bmatrix} 16 & 32 \\ 0 & 16 \end{bmatrix}$

    A^4 is equal to $\left(A^2\right)^2$. By matrix multiplication, $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$. This result squared is $\begin{bmatrix} 16 & 32 \\ 0 & 16 \end{bmatrix}$.

12. (d) 126

    In the binomial expansion of $(x + y)^9$, what is the coefficient of the x^4y^5 term?
    The N^{th} term of an M^{th}-degree binomial expansion is $\binom{M}{M-N} x^{(M-N)} y^N$. $\binom{9}{5} = \frac{9!}{5!4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126$

13. (d) 500

The number of zero-digits at the end of (2009!) will be decided by the number of 2's and 5's which are factors of (2009!). With quick consideration, there will be fewer 5's than 2's. Less than 2009, there are 401 multiples of 5, 80 multiples of 25, 16 multiples of 125, and 3 multiples of 625. Thus, there are will be (401 + 80 + 16 + 3) = 500 factors of 5, and at least 500 factors of 2 which divide (2009!).

14. (c) 640

Tide speed = X. Speed with tide = (100 + X). Speed against tide = (100 – X). 4(100 + X) = 16(100 – X). Solving for X gives that X = 60. Swimming 4 minutes at 160 yards per minute will move Greg 640 yards.

15. (c) – 2c

Since $\csc x = (\sin x)^{-1}$, the expression $\log_a(\csc^2 x)$ is equal to (-2) $\log_a(\sin x)$ = (-2) · c.

16. (b) $\frac{\sqrt{5}+1}{2}$

Examine the complex fraction as a matter of repetition. $B = 1 + \frac{B}{1+B}$; by clearing denominators, you get B(1 + B) = (1 + B) + B. Put into standard form, this is the quadratic equation $B^2 - B + 1 = 0$. Solving for B gives $\frac{1 \pm \sqrt{5}}{2}$. However, since B is clearly greater than one, the answer must be $\frac{1+\sqrt{5}}{2}$.

17. (b) 5

The Remainder Theorem states that if a function, f(x) is divided by (x – c), the remainder is f(c). Evaluating the expression at x = -1, we get 13 – (-6) + (-12) – (-3) – 5 = 5.

18. (e) -10

If the determinant of A is equal to 0, then A^{-1} does not exist. The determinant of A is ($x^2 + 10x - 24$), which is equal to zero when x = -12 or 2. The sum of these results is -10.

19. (a) 8

Using common bases, $7^y = 7^{2x+12}$ and $2^{5x} = 2^{4y-36}$. So, y = 2x + 12, and 5x = 4y – 36. Solving by substitution, x = -4 and y = 4. The difference between x and y is 8.

20. (a) $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

For complex number z = a + bi, the equivalent value in the complex plane is |z| · (cos θ + i sin θ) = |z| · $e^{i\theta}$. As |z| = 1 for all choices, and as we need $z^6 = -1$, then $\left(e^{i\theta}\right)^6 = -1$. The corresponding θ's to the five choices are $\frac{-\pi}{6}$, $\frac{\pi}{3}$, $\frac{-\pi}{4}$, $\frac{\pi}{4}$, and $\frac{-\pi}{3}$. The only θ which will satisfy $e^{6i\theta} = -1$ is $\frac{-\pi}{6}$.

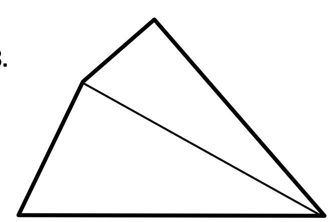
21. (d) 3731

$3G^2 + 5G + 9 = 2009$, or $3G^2 = -5G + 2000$. Since 5 divides -5G + 2000, it must divide $3G^2$. So, G must be a multiple of 5. Checking small multiples of 5 for $3G^2 + 5G = 2000$, G = 25 is a quick solution. So, it must be the case that M = 8. Since $8^0 = 1$, $8^1 = 8$, $8^2 = 64$, $8^3 = 512$, it is easy to see that 8^3 goes into 2009 no more than 3 times, so the answer must be a four-digit number starting with 3. Checking, 2009 is, indeed, equal to 3 · 512 + 7 · 64 + 3 · 8 + 1.

22. (e) $\frac{4}{21}$

Given that each brother rolls either an odd or an even number, there would be $2^6 = 64$ outcomes from the result of their rolls. However, since it is known that there are at least 3 even numbers, then the number of ways that exactly 3, 4, 5, or 6 evens can be rolled is found by adding $\binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 20 + 15 + 6 + 1 = 42$. Given that exactly 3, 4, 5, or 6 evens are rolled, the number of ways that Largo, Milo, and Paulo can possess three of those evens is found by calculating whom, of Kojo, Nico, and Otto, does not have an even number, adding $\binom{3}{3} + \binom{3}{2} + \binom{3}{1} + \binom{3}{0} = 1 + 3 + 3 + 1 = 8$. Thus, in 8 of the possible 42 scenarios in which there are at least 3 even rolls, Largo, Milo, and Paulo all have even rolls.

23. (b) $30 + 2\sqrt{510}$



By the Pythagorean Theorem, the shown diagonal must have length equal to 13. Since the height and base of the right triangle are known (with a little rotation), the area of the right triangle is 30 square units. The area of the other triangle is calculated by Heron's formula. Let p be the perimeter of a triangle. Let s be half of p (call it the semi-perimeter or the triangle). Heron's formula states that, for any triangle with sides a, b, c, the area is calculated by the following formula:

$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{17 \cdot 3 \cdot 4 \cdot 10} = 2\sqrt{510}$.

24. (b) 193

Let X be the amount of time after 3 PM when the minute hand crosses the hour hand, measured in minutes. The equation to find X is: $X = 15 + \frac{X}{60} \cdot 5$. Solving, $X = \frac{180}{11}$, or $16\frac{4}{11}$, or about 16 minutes, 22 seconds. The second hand overlaps the minute hand 59 times per hour. From noon to 3 o'clock, that makes 177 overlaps. From 3PM to 3:16 PM, the second hand will overlap the minute hand 15 more times. At 3:16:22, the second hand will have passed over the minute hand one additional time (at about 3:16:16).

25. (e) $\frac{1000 \cos 48}{\cos 66}$

Because of the nature of supplementary angles, the obtuse angle opposite M measures 138°. By the Law of Sines, $\frac{\sin 138}{M} = \frac{\sin 24}{1000}$. Solving for M, we get that $M = \frac{1000 \sin 138}{\sin 24}$. Now, trigonometric identities verify that $\sin \Theta = \sin (180 - \Theta)$, and that $\sin \Theta = \cos (90 - \Theta)$, where $0 \leq \Theta \leq 90$. Using these properties, $\sin 138 = \sin 42 = \cos 48$, and $\sin 24 = \cos 66$. By replacement, we get $\frac{1000 \cos 48}{\cos 66}$.

26. (c) 410

Let A be the smallest member of the sequence, and let D be the difference between consecutive terms. In an arithmetic sequence, the mean and median are the same number, so the 4th term is equal to one seventh of 2009, or 287. We have that $A + 3D = 287$, and $A + (A + D) = \frac{1}{3}(A + 4D + A + 5D + A + 6D)$. Solving the system of equations, we get that $A = 164$ and $D = 41$. Thus, the largest element of the sequence, $(A + 6D) = 164 + 6 \cdot 41 = 410$.

27. (a) 0 to 0

From the tournament points, NE must have had 2 wins and 1 tie. LA must have had 2 losses and 1 tie. NE and LA could not have tied each other, because NE must have scored at least 2 goals against LA (since at most 3 goals were scored against LA by DC and RSL). Since NE and LA did not tie each other, they must have been involved in ties with DC and/or RSL. If both NE and LA tied DC, then the 3 NE matches would have accounted for 4 of the goals, the 2 remaining DC matches would have accounted for 3 of the goals, and the RSL vs LA match would involved 2 goals being scored, which creates a contradiction at 2-0, 1-1, or 0-2. If both NE and LA tied RSL, then DC would have to have had 2 wins and 1 loess, with scores 0-1, 1-0, and 1-0; however, this would force a situation of 3 goals being scored in the tie match between RSL and LA, thus creating a contradiction. So, DC and RSL must have each tied one of NE and LA. If DC had any ties, they had two ties and one win. Therefore, they must have tied NE, since they did not beat NE. Similarly, if RSL had any ties, they had two ties and one loss. Therefore, they must have tied LA, since they did not lose to LA. It follows that DC and RSL must have tied each other. So, it must be the case that NE tied DC and beat RSL and LA, DC tied RSL and beat LA, and RSL tied LA. The 3 NE matches involved a total of 4 goals being scored (since none were scored against NE). The remaining two DC matches involved 3 goals being scored (since DC tied NE, 0 to 0). The only remaining match, a tie between RSL and LA, had the only other 2 goals, and was thus tied 1 to 1. Since RSL's lone goal was against LA, and since DC tied RSL, the DC vs RSL match ended 0 to 0.

28. (c) $\frac{3}{8}$

Let M be a miss and H be a hit. The outcomes which would result in Lara winning would be MH, MMMH, MMMMMH… in other words, Lara will win if the first hit occurs on an even-numbered shot. $P(MH) = \frac{3}{5} \cdot \frac{2}{5}$; $P(MMMH) = \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}$; $P(MMMMMH) = \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}$; this creates a geometric series with first term $A = \frac{6}{25}$ and common ratio $R = \frac{9}{25}$. The sum of an infinite geometric series, where R is between 0 and 1, is $\frac{A}{1 - R}$, or $\frac{\frac{6}{25}}{1 - \frac{9}{25}} = \frac{3}{8}$.

29. (e) $\frac{4}{9}$

This is a layered geometric series. There is one series with first term $A = \frac{1}{4}$, common ratio $R = \frac{1}{4}$, added to a series with first term $A = \frac{1}{16}$, common ratio $R = \frac{1}{4}$, added to a series with first term $A = \frac{1}{64}$, common ratio $R = \frac{1}{4}$, and so on. Since the sum of an infinite geometric series, where R is between 0 and 1, is $\frac{A}{1 - R}$, then the sums of these individual series are $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}$, … which itself represents a geometric series which totals to $\frac{4}{9}$.

30. (c) 6

Let $f(x) = (x - a)(x - b)(x - c)(x - d)$, where a, b, c, and d are the four roots of the polynomial. Multiplying these four factors out, the coefficient of the 3rd degree term would be $-(a + b + c + d)$. From the function itself, this sum is equal to -6. So, $(a + b + c + d) = 6$.

31. (d) $\frac{3}{16}$

The probability that a crew member is an electrician with a degree is $(0.2)(0.6) = 0.12$. By the same reasoning, the probability that a crew member is a plumber with a degree is 0.12, and the probability that a crew member is an engineer with a degree is 0.40. The probability that a crew member has a degree, then, is $0.12 + 0.12 + 0.40 = 0.64$. The probability that a crew member with a degree is an electrician is $\frac{0.12}{0.64} = \frac{3}{16}$

32. (b) 95

First, the foundation of the problem is that the two shortest legs of a triangle must have a sum greater than the longest leg. The pole of length 1 cannot be used at all for this reason. Let X be the length of the smallest pole in your triangle, and Y be the difference between the lengths of the middle and longest poles. For example, the triangle made with pole lengths 4, 8, 10 would have (X, Y) = (4, 2). X must be no greater than 10. The chart values indicate the number of triangles which can be created for a given (X, Y). The sum of all possible triangles is 95.

	X = 2	X = 3	X = 4	X = 5	X = 6	X = 7	X = 8	X = 9	X = 10
Y = 1	9	8	7	6	5	4	3	2	1
Y = 2		7	6	5	4	3	2	1	
Y = 3			5	4	3	2	1		
Y = 4				3	2	1			
Y = 5					1				

33. (e) 0

If $\frac{2y}{9x} = \frac{x}{2y}$; cross-multiplying, $9x^2 = 4y^2$. In standard form, $9x^2 - 4y^2 = 0$, or $(3x - 2y)(3x + 2y) = 0$. One of these factors must be equal to 0. Since $\frac{2y}{9x} = \frac{x}{2y} = (12x - 8y) = 4(3x - 2y)$, none of these factors can be equal to 0. Thus, $(3x + 2y) = 0$. By factoring, $(9x + 6y) = 3(3x + 2y) = 0$.

34. (a) I

If a new data point is higher than the old mean, the mean must rise, even if only a small amount. The median, on the other hand, may or may not rise, depending on the particular values in the middle of the data. The range will not necessarily increase, as the new data point may or may not be the new highest data point. The mean and the median can, given the right set of data, increase exactly the same amount. For example, let the original data be {0, 4, 8, 12, ... 64, 68, 72, 76}, which has mean 38 and median 38. By adding 80 to the data set, the mean and median shift to 40.

35. (a) $\frac{40 - 9\pi}{40}$

$\text{Cot } x = \frac{\cos x}{\sin x} \geq 1$. For the given span, this ratio will be greater than or equal to 1 when x is in the ranges of $[0, \frac{\pi}{4}]$, $[\pi, \frac{5\pi}{4}]$, $[2\pi, \frac{9\pi}{4}]$, and $[3\pi, 10]$. The sum of these ranges is $\frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} + (10 - 3\pi) = 10 - \frac{9\pi}{4}$. Dividing this total by 10 gives (a) $\frac{40 - 9\pi}{40}$.

36. (d) $\frac{\pi}{8}$

Substituting $u = x^2$, $du = 2x\,dx$, you get $\lim_{b\to\infty} \frac{1}{2}\int_1^b \frac{1}{u^2+1}\,du = \frac{1}{2}(\tan^{-1} b - \tan^{-1} 1) = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$.

37. (b) II only

$M^5 = 5M$. Since M is positive, $M^4 = 5$, and is thus, rational. $(M^4)^3 = M^{12}$ is also, necessarily, rational. The others are not.

38. (c) $\frac{135}{2}$

By completing both squares, $2\left(x - \frac{9}{2}\right)^2 + 3(y - 3)^2 = \frac{135}{2} - B$. Since both squares are non-negative, B cannot be any smaller.

39. (b) $\frac{-\sqrt{3}}{3}$

The slope of a polar curve at (r, ϴ) = $\frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$. Using that r' = - cos ϴ, $\sin\frac{\pi}{6} = \frac{1}{2}$, and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, then the aforementioned ratio simplifies to $\frac{-\sqrt{3}}{3}$.

40. (d) Green

(I) implies that if a player has a yellow card, then she also has an orange card. (II) implies that if a player has a blue card, then she also has a yellow and a red card. (IV) implies that if a player has an orange card, then she also has a blue card. (V) implies that each player must have at least one of orange, yellow, and blue. Since yellow implies orange, orange implies blue, and blue implies yellow, then every player must have yellow, orange and blue. Since each player has blue, each player also has red. Thus, all players have red, orange, yellow and blue. If one player is the only player holding all five colors, then that player uniquely hold green, as everybody else has all other colors. (III) implies only that each player has yellow and/or green.